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Decision Impact of Stochastic Price Models in the Petroleum Industry

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Decision Impact of Stochastic Price Models in the Petroleum Industry

by

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Dedication

To my parents.

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Abstract

Decision Impact of Stochastic Price Models in the Petroleum Industry

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The University of Texas at Austin, 2011

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Stochastic price models have proven material to decision making in the oil industry when accurate valuations are important, but little consideration is given to their impact on decisions based on relative project rankings. Traditional industry economic analysis methods do not usually consider uncertainty in oil price, although the real options literature has shown that this practice underestimates the value of projects that have flexibility. Monetary budget constraints are not always the limiting constraints in decision making; there may be other constraints that limit the number of projects a company can undertake. We consider building a portfolio of upstream petroleum development projects to determine the relevance of stochastic price models to a decision for which accurate valuations may not be important. The results provide guidelines to determine when a stochastic price model should be used in economic analysis of petroleum projects.

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1. Introduction

The upstream oil and gas industry is characterized by its high uncertainty and potentially large rewards. As in any industry or venture, sound decision making is crucial to success, and it is important to carefully understand the decision making process when dealing with high risk, high reward decisions. Practitioners and academics alike recognize that uncertainty is inherent throughout the process of finding, extracting, refining, and marketing oil and gas. The industry makes considerable use of detailed models of underground reservoirs, utilizing information from exploratory wells and seismic tests that can be expensive and only provide incomplete information. A significant source of uncertainty that is typically not given this level of attention, but is integral to the value of the asset, is the future market price of the oil or gas (Bickel and Bratvold 2008).

High-production projects typically have horizons of several decades. With these long time frames there is significant uncertainty in how the price of oil (or gas) will behave over the life of the project. There are several well-known stochastic models that describe the dynamics of commodity prices, and it might seem prudent to explicitly model the uncertainty in a factor that has significant impact on project value. In particular, if there are future decisions regarding a project that management knows it will face, modeling oil price uncertainty can shed light on the various market scenarios that may unfold, which in turn leads to valuing this future flexibility more accurately. At first glance, these are compelling arguments for modeling oil price stochastically when evaluating development investments. However, practitioners in the oil industry do not generally agree. Their economic analysis does not typically treat market uncertainties rigorously. This is not out of ignorance of the uncertainty, but rather a modeling choice. The simplest model that is sufficient for good decision making is highly desirable. The concern is whether stochastic price models are required for good decision making, or if the simpler techniques generally used are sufficient. Are stochastic price models relevant to decision making in the oil industry? If so, under what conditions? The goal of this work is to address these questions.

There are many different types of decisions regarding project investment that a company might face; we make a general distinction between "valuation" decisions and "ranking" decisions. A valuation decision involves a decision such as setting a buying or selling price. A ranking decision involves making a decision based on the relative rankings of a set of alternatives, such as choosing the best subset of projects to undertake. The real options literature has shown that stochastic models of uncertainty are integral to accurate project valuation, particularly when there are future decisions to be made after some uncertainty has been resolved. This thesis is focused on ranking decisions, for which it is not immediately clear whether good decision making requires stochastic modeling.

To examine the relevance of stochastic price models to ranking decisions, we consider the decision of choosing a subset (a portfolio) of projects from a set of candidate projects. The projects are oil fields that have already been appraised, and if included in the portfolio will be developed to produce oil. We specifically consider crude oil primary-stage production, but the analysis could easily extend to secondary and tertiary production, or to other products such as natural gas. Throughout this paper the term "portfolio" refers simply to the set of projects chosen for development. Project inclusion is binary; i.e. we do not allow fractional investment. We also refer to any future decision, "management flexibility," or "real option" simply as an option.

To compare the quality of decisions made using different models, we compare the portfolios constructed using each of three stochastic models, a fixed price, and a model using forward contract prices for valuation. As described more fully in Chapter 3, the stochastic models we consider are a two-factor model, a mean reverting model, and a geometric Brownian motion model. We assume that the two-factor model is the best description of the three for oil prices. We take it to be the "true" price process, and use it as the standard of comparison for the other models.

One of the primary arguments in the real options literature for using stochastic models (and real options analysis in general) is that discounted cash flow analysis with a fixed price does not correctly value flexibility (i.e., the inclusion of options) in a project. We examine this argument in the setting of ranking decisions by considering both the case when projects have options and when they do not.

The results indicate that for many ranking-decision situations, stochastic price models do not add significant value to analysis. They can, however, be relevant for ranking decisions where accurate valuation is important, such as when a project may be excluded for having negative value. For a pure ranking-decision, such as choosing from among projects that all have positive value, stochastic price models are not material in the cases we consider.

The layout of the thesis is as follows: Chapter 2 reviews the literature on price models and decision making in the oil industry. Chapter 3 describes the price models we consider and the grid model we use to approximate them. Chapter 4 describes the project model, valuation of projects and options using the grid approximation, and motivates the metrics we use in our analysis. Chapter 5 details the results of various situations which support our conclusions. Chapter 6 concludes the thesis and summarizes the results and their implications to industry.

2. Literature Review

Due to the characteristics of the oil and gas industry already mentioned, the industry is a prime application area for decision analysis techniques. Much work has been done on this topic, as well as on modeling uncertainty and flexibility in real assets. However, due to the focus of this research, we limit our literature review to the area of stochastic price modeling.

2.1. Price Models

In our analysis we consider three stochastic price models: Geometric Brownian Motion (GBM), a Mean Reversion (MR) model, and a two-factor (TF) model (Schwartz and Smith 2000). As we discuss below, all three have been applied to the analysis of oil production projects in the literature.

Luenberger (1998) provides a good introduction to GBM, the simplest of the stochastic models we consider. It is the classic model of stock prices and originated from physics as a model of particle movement. Many models of market uncertainty begin with this model. It is often used as the model for uncertainty in real options problems, but is recognized to be a poor price model for consumption commodities such as oil or gas.

Baker, Mayfield, and Parson (1998) discuss and compare three models of commodity prices, specifically oil prices. The models include GBM, MR, and a two-factor mean reverting model with an uncertain mean described by a random walk. This third model is identical to the model presented by Schwartz and Smith (2000). The one-factor model Baker, Mayfield, and Parson (1998) describe is identical to the mean reverting model we use. In particular, its process is a special case of the two-factor model that has drift, but no uncertainty, in the long-term mean. Several other mean reverting models are described as interest-rate models in Luenberger (1998) and Hull (2000), but none of these allow for drift in the long-term mean. Baker, Mayfield, and Parson (1998) also discuss parameter estimation for each of the models and how well they describe the prices of futures contracts going forward from the time of estimation. They found that TF best forecasts futures prices out of the three, and that adding a third factor to consider uncertainty in the interest rate does not significantly improve forecasts. Schwartz and Smith (2000) comprehensively describe the TF model that describes both short-term price deviations and dynamics of the long-term mean. In addition to MR, GBM is a special case of this model. We take TF as our "gold standard" and use it as a point of comparison for the other stochastic models and the deterministic treatments of oil price.

In the literature, parameter estimation for models with mean reversion uses either estimation from historical data or current futures prices, which encode the current market belief about future prices. GBM model parameters can be estimated from historical oil price data. Hull (2000) provides a proof that when interest rates are constant (as we assume), that forward contract prices and futures prices are equal. One way to handle market price information in a deterministic "model" is to use the current forward contract prices for a series of contract durations as the deterministic future prices.

Al-Harthy (2007) compares GBM, MR, and MR with jumps regarding the value of an offshore well with no production uncertainties. He focuses on the effects of oil price uncertainty on project value uncertainty. His findings show that variance has more impact on project value for the GBM model than the MR model. The MR model results are determined mainly by the initial price and long-term price, and impacted very little by price volatility. The MR with jumps model is sensitive to these parameters, but also to the number of jumps and to a larger extent their size. His results show that both project value and the uncertainty of that value are dependent upon the price model used. His analysis considers the value of a single project, crucial for a valuation-decision, but does not examine ranking-decisions. He also does not consider any options in the project model he uses for analysis. We consider input price uncertainty on ranking-decisions both with and without options. We do not consider the MR with jumps model, as the jump process is not a factor in the TF model.

Oftedal (2008) looks at the differences in the rankings of ten projects to compare three price models: a constant price, the expected prices from a mean reverting process, and a simulated stochastic mean reverting process. He analyzed each model using three different valuation metrics: net present value (NPV), NPV divided by capital expenditure, and internal rate of return (IRR). He concluded that both project valuations and rankings are affected by both the choice of price model and the valuation metric. In his research, only one candidate set of 10 projects is considered. It is not clear whether his results represent general effects of price models or are particular to those specific projects. He does not consider the TF model, portfolio values using the different models, or any significant flexibility. Although the project rankings compared across different models only differ by at most one rank for all but one project under each valuation method, he concludes that using stochastic price models significantly effects rankings. No investigation of the impact (such as on portfolio value) of these small differences is mentioned. We expand upon this work by considering multiple stochastic price models, the differences in decisions made using each, and the value impacts of these differences over a wide range of possible project candidate sets.

2.2. Decision Making in the Petroleum Industry

The methods referred to as real options analysis are often billed as replacements for discounted cash flow (DCF). Laughton (1998a) describes several criticisms of using DCF methods for financial analysis of petroleum projects. DCF can bias the analysis by underestimating the value of managerial flexibility. It is very sensitive to the discount or hurdle rate used, which is often chosen without regard for specific project risks. Risk is sometimes naively, and incorrectly, incorporated by heuristically adjusting this discount rate. He introduces the Modern Asset Pricing (MAP) approach as an integrated and rigorous alternative to DCF. Salahor (1998), Bradley (1998), Laughton (1998b) and Laughton (1998c) further detail the MAP approach.

Smith and McCardle (1999) discuss option pricing methods and their relation to decision-analysis methods. These methods are "equally capable of modeling flexibility." They describe applying these methods to case studies of real-world problems. They solve a multi-phase petroleum development project with several embedded options as a dynamic program, using MR to model oil price. They identify two major issues in applying option pricing methods to projects (or "real" assets): modeling flexibilities and valuation of cash flows. The standard practice at the company they partnered with on the study (Chevron) was to analyze projects using "low", "medium", and "high" fixed prices and to discount future cash flows using a risk adjusted discount rate.

Brandão, Dyer, and Hahn (2005a) describe valuing real options with binomial decision trees rather than the binomial lattices commonly used in finance methods. Their approach is a three-step process, which first values a project without options with the DCF method to obtain the project's present expected value. The volatility of the project is then determined by simulating the various uncertainties that affect project value. Finally, a binomial tree is constructed to approximate a GBM process with an initial value of the present expected value determined in the first step and standard deviation determined as the volatility of the project in the second step. Although in their oil project valuation example they use GBM model for oil price, they explain that other stochastic price models can be used. Smith (2005) compares their three-step approach to an alternative “fully risk-neutral” method that uses a single model with risk neutral probabilities and discounts at the risk free rate to value projects both with and without options. He points out the differences between the two approaches, which Brandão, Dyer, and Hahn (2005b) clarify are primarily related to modeling preferences, and that both yield the same results.

Bickel and Bratvold (2008) describe the results of a survey of almost 500 oil and gas professionals regarding uncertainty quantification and decision making in the industry. They found that “while hydrocarbon prices are an important source of uncertainty (the second-highest-ranked uncertainty [as ranked “important or significant” in 78% of responses]), there is relatively little support for increasing the level of probabilistic modeling to capture this [as evident by only 29% of responses saying hydrocarbon prices warranted 'more than minor improvements'].” They suggest that “this response may stem from a realization that decision making is what counts and improving uncertainty quantification [or valuation] may not improve decision making.” Uncertainty quantification is only useful if it can change a decision. Al-Harthy (2007) also discusses the common practice of deterministically modeling "above ground" uncertainties, particularly oil price. Thus, it seems that while academics have advocated the use of stochastic price models, the industry has not been quick to accept this guidance. As one practitioner lamented at a recent SPE forum on uncertainty quantification, “Price moves all projects in the same way.”[1]

Smith and Nau (1995) explain that both option pricing methods and decision analysis methods, when used correctly, give consistent results in evaluating a project that may have both market risk and private risk. They describe an approach that uses "risk neutral" probabilities derived from market data and expected values for the market risks,

[1] Personal communication with Professor J. Eric Bickel (March 2011).

and assessed "subjective" probabilities and certainty equivalents using a utility function for the private risks. This method is the basis for the “fully risk-neutral” method Smith (2005) describes. We use this valuation method to value projects with and without options, using risk neutral probabilities determined from market data, and discounting at the risk free rate. As Smith and Nau (1995) show, this method is consistent with the other methods described, and so our choice of the valuation method does not affect the results.

3. Price Models

Typically, the decision to invest in a particular project is based on an analysis and valuation using a single oil price that is fixed throughout the life of the project. Practitioners know this is not realistic, but do not perceive value in modeling price stochastically. It should be noted that the purpose of stochastically modeling price is not to predict future prices but to inform a decision that depends upon price.

To isolate the decision impact of stochastic price models, we do not consider any sources of uncertainty other than oil price. We make the reasonable assumption that the market for oil is complete, meaning we can perfectly hedge any risk due to oil price by buying and selling publicly traded securities. Then according to Smith and Nau (1995), since the projects have no private risks and only market risks, we value projects using risk neutral probabilities and expected values, discounting at the risk-free rate.

3.1. Stochastic Price Models

3.1.1. Geometric Brownian Motion

Geometric Brownian Motion (GBM), also referred to as a random walk model, is the simplest of the three stochastic process models. GBM assumes that the price at time $t + \Delta t$ is dependent only on the price at time t. There is a deterministic drift component which causes the price to rise or fall on average. If the drift component is zero, the expected price change is zero and all movement is due to the random increment, dz. The price process is given by

$$d \ln S_t = \mu_\xi dt + \sigma_\xi dz \,, \tag{1}$$

where μ_ξ is the mean deviation or drift, σ_ξ is the standard deviation, and dz is an increment of a standard Wiener process, which is normally distributed with a mean of zero and standard deviation of one. At any time t, the log of price is given by

$$\ln S_t = \ln S_0 + \mu_\xi t + \sigma_\xi z_t . \tag{2}$$

The expected price at time *t* is given by

$$\mathrm{E}\left[S_t\right] = \exp\left(\xi_0 + \left(\mu_\xi + \tfrac{1}{2}\sigma_\xi^2\right)t\right). \tag{3}$$

Figure 1 shows the mean, P10, P50, and P90 percentiles and a single randomly generated sample path of the GBM process for oil price over 30 years beginning in 2008 with increments of 2 months. The mean and variance of the process are those parameters for the long-term factor of the TF model, described below, estimated from the prices of futures contracts in August 2010 for delivery dates between October 2010 and December 2018[2].

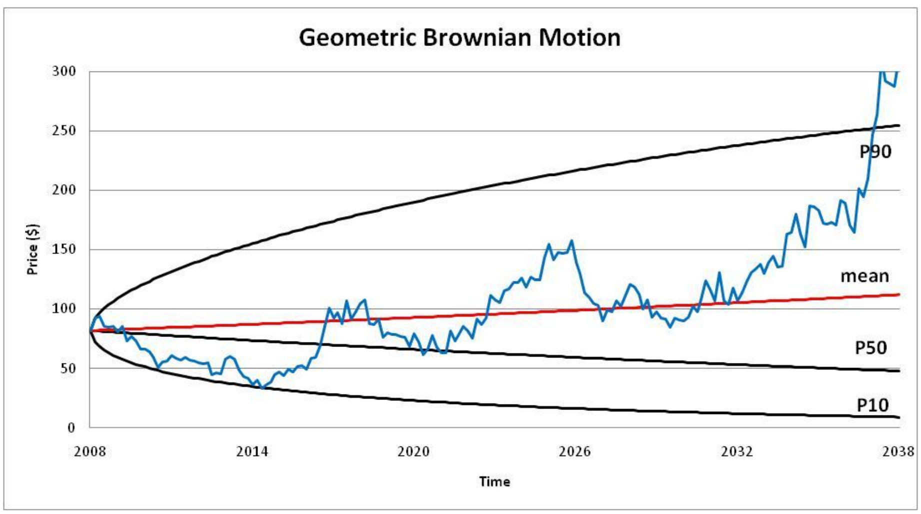

Figure 1: Geometric Brownian Motion for oil price: mean, P10, P50, P90, and one sample path.

3.1.2. Mean Reversion

Mean reverting processes are often used to model consumption commodities (e.g., oil, natural gas, copper, wheat, etc.) and interest rates. Consumption commodities, unlike

[2] Parameter estimates provided by Jim Smith.

stock prices, are subject to supply and demand market forces. When prices drop below the long-term mean production is decreased to reduce costs, which decreases supply and increases demand, eventually driving prices back towards the mean. Similarly when prices rise high above the mean, new production might be brought online thus increasing supply and driving prices down.

The particular MR model we use, described in Baker, Mayfield, and Parsons (1998), incorporates long-term market beliefs with a deterministically drifting mean. Changes in the log of the price are characterized by an Ornstein-Uhlenbeck process. The process is described by an uncertain short-term factor χ_t and a deterministic long-term factor ξ_t which are related to price by

$$\ln(S_t) = \chi_t + \xi_t, \tag{4}$$

where increments of the factors are given by

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \tag{5}$$

$$d\xi_t = \left(\mu_\xi + \tfrac{1}{2}\sigma_\xi^2\right)dt. \tag{6}$$

The $\frac{1}{2}\sigma_\xi^2$ term in equation (6) is included because μ_ξ and σ_ξ are parameters for a lognormal distribution, which has a mean of $\mu_\xi + \frac{1}{2}\sigma_\xi^2$. The expected price at time t is given by

$$\mathrm{E}[S_t] = \exp\left(e^{-\kappa t}\chi_0 + \xi_0 + \left(\mu_\xi + \tfrac{1}{2}\sigma_\xi^2\right)t + \tfrac{1}{2}\left(\left(1-e^{-2\kappa t}\right)\frac{\sigma_\chi^2}{2\kappa}\right)\right). \tag{7}$$

Figure 2 shows the mean, P10, P50, and P90 percentiles and a single randomly generated sample path of the MR process for oil price over 30 years beginning in 2008 with increments of 2 months. Like GBM above, the parameters for MR are the mean from the long-term factor for TF for the deterministic drift, and the short-term parameters for the mean reversion.

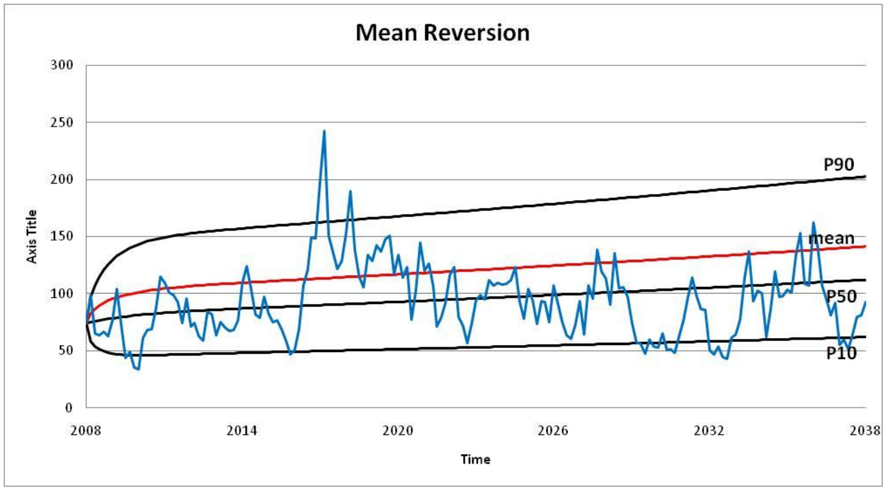

Figure 2: Mean Reversion for oil price: mean, P10, P50, P90, and one sample path.

3.1.3. Two-Factor

The Schwartz-Smith (2000) TF model models price as a combination of an Ornstein-Uhlenbeck process for short-term deviations and a Brownian Motion process for changes in the long-term mean. This is described by a model similar to MR, except that the drift is also uncertain. The log of price for TF is given by

$$\ln(S_t) = \chi_t + \xi_t \tag{8}$$

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \tag{9}$$

$$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi \tag{10}$$

where the increments of the two processes are correlated according to $dz_\chi dz_\xi = \rho_{\chi\xi} dt$.

The expected price is given by

$$\mathrm{E}[S_t] = \exp\left(e^{-\kappa t}\chi_0 + \xi_0 + \mu_\xi t + \tfrac{1}{2}\left(\left(1-e^{-2\kappa t}\right)\frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 t + 2\left(1-e^{-\kappa t}\right)\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} \right) \right) \tag{11}$$

and the parameters are described in Table 1.

Symbol	Description
κ	Short-term mean-reversion rate
σ_χ	Short-term volatility
dz_χ	Short-term process increments
μ_ξ	Long-term (equilibrium) drift rate
σ_ξ	Long-term (equilibrium) volatility
dz_ξ	Long-term (equilibrium) Process Increments
$\rho_{\chi\xi}$	Correlation in Increments

Table 1: Two-factor model parameters.

To generate the TF process for a desired time step Δt, we generate two standard normal variates z_χ and z_ξ. These variates are correlated according to the $\rho_{\chi\xi}$ term. Then, using the difference equations

$$\chi_{t+\Delta t} = \chi_t - \kappa\chi_t\Delta t + \sigma_\chi\sqrt{\Delta t}z_\chi \tag{12}$$

$$\xi_{t+\Delta t} = \xi_t + \mu_\xi\Delta t + \sigma_\xi\sqrt{\Delta t}z_\xi\,, \tag{13}$$

the price for time period $t+\Delta t$ is given by

$$S_{t+\Delta t} = \exp\left(\chi_{t+\Delta t} + \xi_{t+\Delta t}\right). \tag{14}$$

Figure 3 shows the mean, P10, P50, and P90 percentiles and a single randomly generated sample path of the TF process for oil price over 30 years beginning in 2008, with increments of 2 months. The sample path was generated using Equations (12) - (14). Likewise, the sample paths for GBM and MR in Figure 1 and Figure 2, respectively, were generated using equivalent time-discretizations of the two processes.

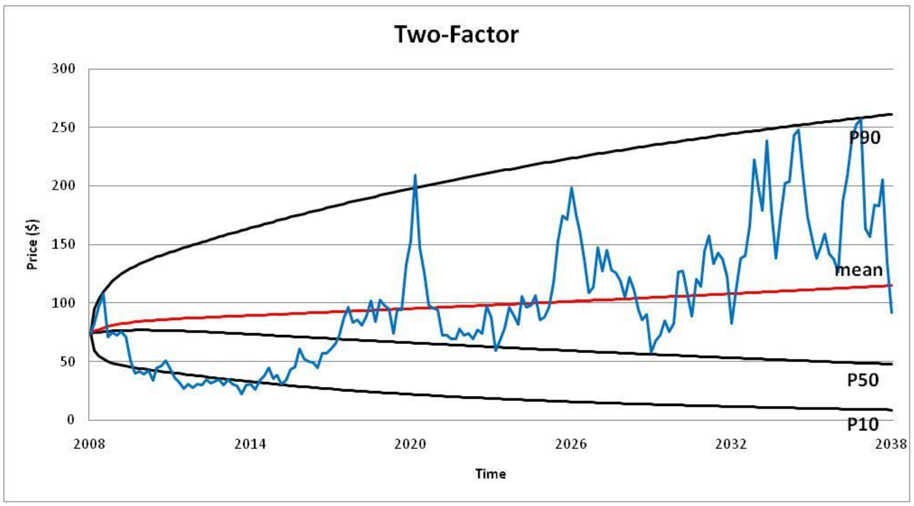

Figure 3: Two-factor model for oil price: mean, P10, P50, P90, and one sample path.

3.1.4. Recovering GBM and MR from TF

Ensuring consistency between the stochastic models we consider is very important. We would like the models to be as close as possible and only differ in their treatment of uncertainty. Several approaches are clearly possible. For example, one could fit GBM, MR, and TF to the same market data and then use these resulting parameters in each of the models. The problem with this approach as far as this thesis is concerned is that differences between the models would be due to both different treatments of uncertainty *and* different fitting procedures. A second approach, and the one taken in this thesis, is to recover the GBM and MR models from the TF model, which nests them. We do this by "zeroing-out" the GBM or the MR components of the TF model. The differences between the results that we discuss later in the thesis are then clearly due to incorporating additional factors in the price model, not a result of different fitting approaches. We now describe our approach in more detail.

We isolate the long-term factor, yielding a GBM model, by setting

$$\chi_0 = 0, \sigma_\chi = 0$$

which makes equation (9) zero at all times, leaving only the long-term factor:

$$d\ln(S_t) = \mu_\xi dt + \sigma_\xi dz_\xi \tag{15}$$

Removing uncertainty in the long-term factor similarly isolates the effects of short-term uncertainty and models the process as a MR model. The long-term uncertainty is removed by setting

$$\sigma_\xi = 0$$

The long-term drift is deterministic, and thus is present in the MR model. The expectation of the long-term factor, which is lognormally distributed with parameters μ_ξ and σ_ξ, is $\mu_\xi + \frac{1}{2}\sigma_\xi^2$. The MR process derived from TF is then

$$d\ln(S_t) = \left(\mu_\xi + \tfrac{1}{2}\sigma_\xi^2\right)dt - \kappa\chi_t dt + \sigma_\chi dz_\chi . \tag{16}$$

3.1.5. Comparison of the Stochastic Models

GBM, MR, and TF each model price under different assumptions. The parameters we use in this section are the same parameters we use for analysis. Figure 4 and Figure 5 show the expected prices and variance, respectively, for each stochastic model. Note that the variance does not change over time, rather it is a function of the length of time over which the price change is considered.

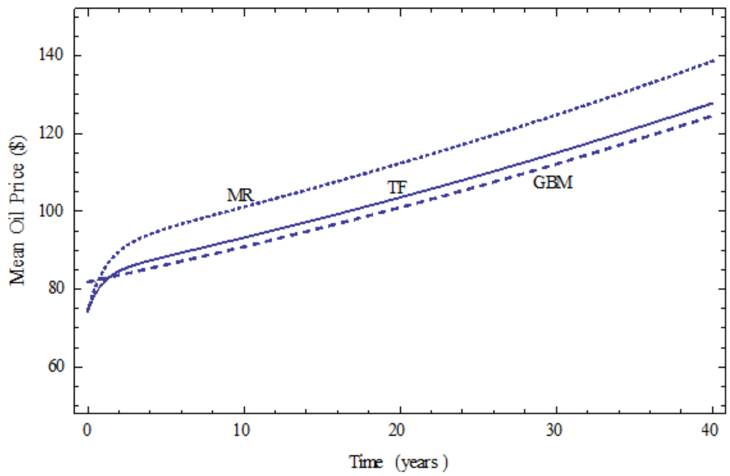

Figure 4: Expected prices for each stochastic model over 40 years.

The expected prices are not identical, but exhibit the same long-term drift after about three years. Each model describes the same drift, but at different levels; MR is greater than TF in the long run and TF is greater than GBM. The variances are not as similar between the models as the expectations. The variances of GBM and TF increase as the time frame increases, and the variance of MR increases quickly to a steady value of $\sigma_{\chi}^{2}(2\kappa)^{-1}$, which is approximately 21.3%. The non-mean reverting factors of both GBM and TF increase their ranges of possible prices over time, which increases our uncertainty about their values. MR has an essentially fixed range that drifts upward over the long run.

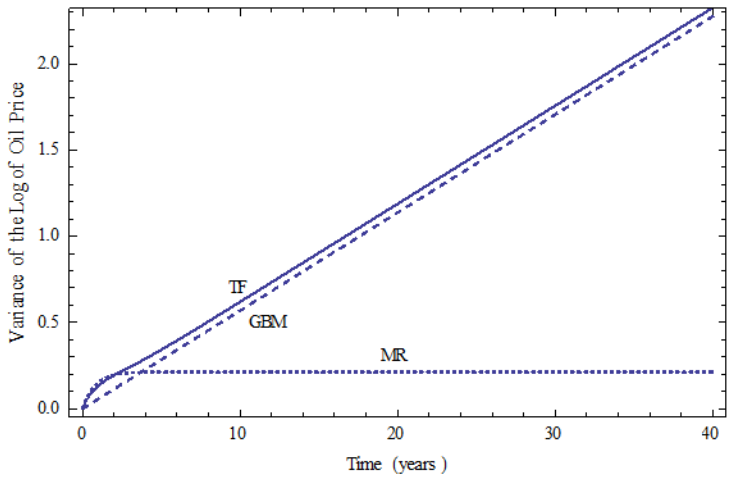

Figure 5: Variance of the log of oil price for each stochastic model over 40 years.

A pure GBM process ignores mean reversion which has been shown to be present in oil prices (Baker, Mayfield, and Parsons 1998). MR ignores uncertainty in the mean to which prices revert. TF captures both factors, but particular decisions may not be sensitive to one or the other.

3.1.6. Estimated Parameters

The model parameters we use were estimated by James Smith using August 2010 futures prices for contracts with delivery dates from October 2010 to December 2018, and are given in Table 2.

The initial equilibrium factor of 4.4048 corresponds to an equilibrium price of \$81.84. Combining this with the initial short-term factor of -0.097 corresponds to an initial spot price of \$74.27. The Kalman-Filtering technique described by Schwartz and Smith (2000) yields these estimates for the TF model parameters.

Symbol	Description	Estimate
κ	Short-term mean-reversion rate	0.6267
σ_χ	Short-term volatility	51.70%
μ_ξ	Equilibrium drift rate	-1.79%
σ_ξ	Equilibrium volatility	23.85%
$\rho_{\chi\xi}$	Correlation in Increments	-0.4136
χ_0	Init Equilibrium Factor	4.4048
ξ_0	Init Deviation Factor	-0.097

Table 2: Estimated values for TF model parameters.

3.2. DETERMINISTIC PRICE MODELS

Deterministic "models" are models in the sense that they contain some information about oil price. We look at two models that do not incorporate any uncertainty: a fixed price and a model comprised of forward strip prices.

3.2.1. Fixed Price

A common practice in the petroleum industry is to use a fixed (F) oil price in DCF calculations to determine net present value. The justification for this approach is that since all projects are affected similarly by oil price, their values will change in tandem with oil price and implicitly their rankings will not change. This is a reasonable conclusion, but may not be valid when oil price is modeled stochastically. We examine this issue in our analysis. The fixed price used in our analysis is $82, which corresponds to the initial long term mean of TF, and is approximately the recent historical average price.

One way to incorporate some uncertainty into the economic model is to consider a discretization of a probability distribution for a fixed price. This model essentially assumes that price is fixed throughout the life of a project, but that this price is uncertain. A common discretization scheme for an uncertainty assumed to have a continuous distribution, normal perhaps, is to take three (or more) percentiles from the cumulative density function and assign each of them a probability. Figure 6 shows the tree representing the discretization, which might be used in a decision tree.

One particular method often used in the oil and gas industry is to weight the 90^{th}, 50^{th}, and 10^{th}percentiles with probabilities 0.25, 0.5, and 0.25, respectively (Bickel, Lake, and Lehman (2011)). We briefly consider this method as an alternative model, but do not treat it thoroughly, as our primary focus is on stochastic process models. In addition, the literature concerning this type of discretization does not address discretization of uncertain quantities that change with time, e.g., oil price. We arbitrarily choose to discretize the distribution of the equilibrium factor of TF at time $t=5$ years, which we denote by $f_{t=5}$, to balance the long time frame of a typical project with the effects of discounting far into the future. Figure 7 shows the discretization of the cumulative density function, $F_{t=5}$. As Bickel, Lake, and Lehman (2011) show, a three point approximation of a lognormal distribution can be quite inaccurate, so we discretize the log of the long-term factor, rather than the distribution of price itself. This leads to P90, P50, P10 values for the log of price of 5.0, 4.32, and 3.63, respectively, which correspond to prices of \$148.23, \$74.84, and \$37.78, respectively.

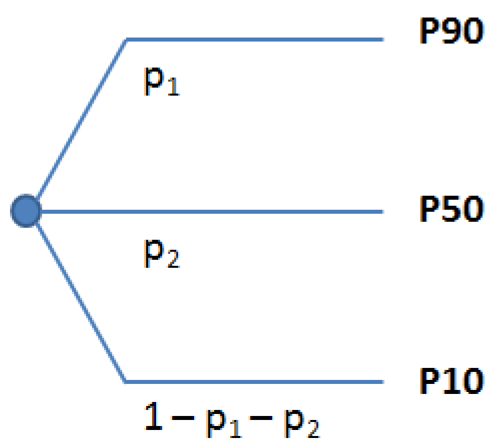

Figure 6: P10, P50, P90 fixed price tree.

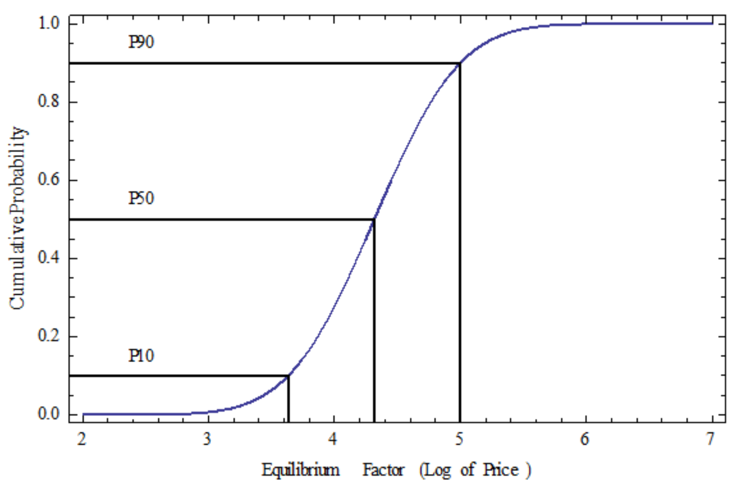

Figure 7: P10, P50, and P90 percentiles of $F_{t=5}$.

3.2.2. Forward Strip

The Forward Strip (FS) model uses the current prices of forward contracts for oil with different delivery dates as estimates of future oil prices. Since the forward contracts typically do not have monthly delivery dates beyond six years from the present, we extrapolate prices beyond that point. The forward prices converge to a long-term trend over this time frame though, and extrapolation over the last year or two of contract maturities gives a long-term price series almost identical to the TF mean. Figure 8 shows the actual forward prices in blue and the extrapolated prices in red.

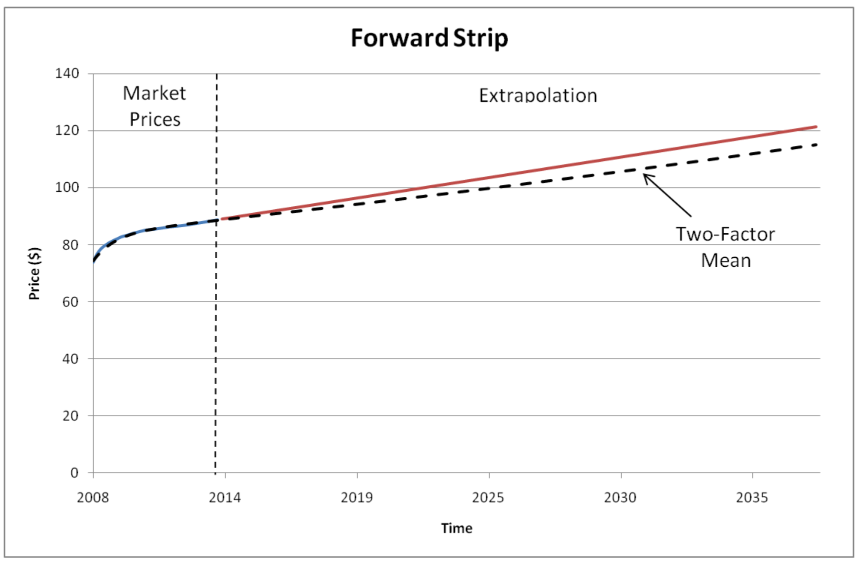

Figure 8: Forward Strip prices (blue) and extrapolated prices (red).

The FS model closely approximates the mean prices of TF because the TF parameters are estimated from futures price data. The FS model lacks the uncertainty in price that TF captures, but is easily obtainable and captures both short-term and long-term mean dynamics.

3.3. Grid Approximation for Stochastic Models

The general idea of grid approximations of stochastic models is to discretize the continuum of oil prices into a finite set of prices represented as states in the grid. This approach generalizes the binomial lattice by allowing the price process to move from any state to any other state in one step with some specified probability. This gives the multinomial lattice a higher fidelity than the binomial lattice. Another advantage of the grid is that once the matrix of transition probabilities is calculated, the same matrix can be used with any initial state. The transition probabilities do not change with time (they are time-homogenous), and are only dependent on the current state. Thus for any state in the state space and any time period, we have the one-step transition probabilities, and we can choose any state as the initial state using the same transition probability matrix. Figure 9 shows a grid with n prices (states) and t time periods. The $n \times n$ transition probability matrix defines the probability of moving from each state to every other state in one time step of length Δt. We use the grid to value both the projects themselves, and the options we consider, which derive their value from those projects. Any option that can be valued with a binomial lattice can be valued using the grid approximation. In our analysis we consider options that Smith (2005) calls scale options, which allow the option holder to scale future cash flows by some predetermined factor at a fixed cost. In §3.3.1 we describe the procedure for calculating the grid transition probabilities for TF, in §3.3.2 we present and justify the grid parameters we use, and in §3.3.3 we describe how to value projects and options using the grid.

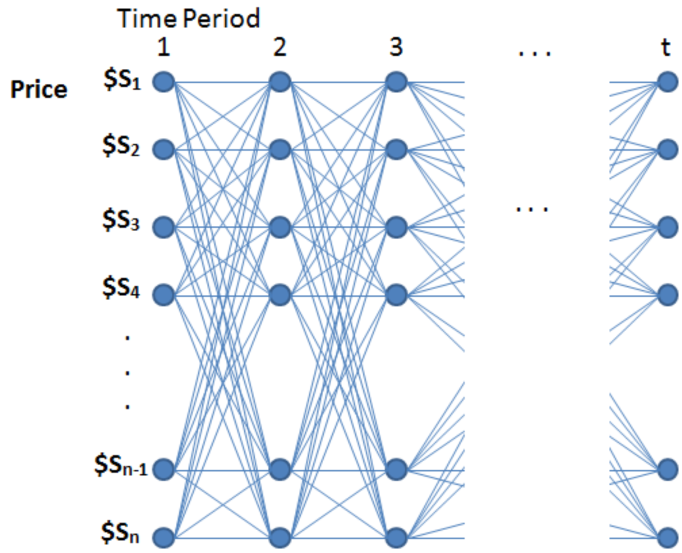

Figure 9: Multinomial Lattice with *n* price states and *t* time periods.

3.3.1. Computing Grid Transition Probabilities

TF models price as a stochastic process over the positive real numbers, $\mathbb{R}_+$, and the grid approximates the process over a finite discretization of the reals. The discretization of price is actually achieved by discretizing the two underlying processes of TF, χ_t and ξ_t. Let $\mathrm{X} \subseteq \mathbb{R}$ and $\Xi \subseteq \mathbb{R}_+$ be the set of discretized values for χ_t and ξ_t, respectively, where for any $\psi \in \mathrm{X}$ and $\omega \in \Xi$, $\psi + \omega \geq 0$. The long-term factor is discretized into $|\Xi| = n_\xi$ values on the range $\left[v_\xi^l, v_\xi^u\right]$ and the short-term-factor is discretized into $|\mathrm{X}| = n_\chi$ values on the interval $\left[v_\chi^l, v_\chi^u\right]$. Let $S \subset \mathbb{R}_+$ be the set of prices in the discretization. Then $f : \mathrm{X} \times \Xi \to S$, where $f(x, y) = e^{x+y}$, is the function relating the values of the short-term and long-term processes of TF to the modeled price. Multiple combinations of short-term deviations and long-term values can correspond to the same price (*f* is surjective). By considering the current spot price of oil without also considering the price history, it is impossible to know whether that price is above or below the prevailing mean price. The states of X and Ξ over τ time periods can either be considered separately in a $|\mathrm{X}|\times|\Xi|\times\tau$ three-dimensional grid, or combined into an aggregate state $\Sigma_{\xi\chi}$ and modeled in a $|\Sigma|\times\tau$ two-dimensional grid. The set of aggregate states, Σ, is every possible pair of states of X and Ξ, or $\Sigma=\mathrm{X}\times\Xi$. We take the second approach as a matter of preference.

Discretizing the two processes into $|\Xi|=n_\xi$ and $|\mathrm{X}|=n_\chi$ state values, and bounding the range of the Ξ states to $\left[v_\xi^l, v_\xi^u\right]$ and the range of the X states to $\left[v_\xi^l, v_\xi^u\right]$, produces state values for the two processes given by

$$\Xi_i = v_\xi^l + (i-1)\frac{v_\xi^u - v_\xi^l}{n_\xi - 1}, i=1,2,\ldots,n_\xi \tag{17}$$

and

$$\mathrm{X}_j = v_\chi^l + (j-1)\frac{v_\chi^u - v_\chi^l}{n_\chi - 1}, j=1,2,\ldots,n_\chi . \tag{18}$$

The transition probability from one state to another in the discretization is really the probability of transitioning from a state i to within a *range* about state *j*; the probability of a point value of a continuously distributed quantity is zero. We now define these ranges as bins of equal size centered on each state value by defining the upper and lower cutoff values for each bin. V_ξ^u is the set of upper cutoff values and V_ξ^l the set of lower cutoff values for the bins of ξ_t, and V_χ^u and V_χ^l the upper and lower cutoffs for the bins of χ_t, respectively. The bin cutoff values for ξ_t are given by

$$V_{\xi,i}^u = \begin{cases} \dfrac{\Xi_{i+1}-\Xi_i}{2} & i=1,2,\ldots,n_\xi-1 \\ +\infty & i=n_\xi \end{cases} \tag{19}$$

$$V_{\xi,i}^l = \begin{cases} -\infty & i=1 \\ \dfrac{\Xi_i-\Xi_{i-1}}{2} & i=2,3,\ldots,n_\xi-1 \end{cases} \tag{20}$$

and for χ_t,

$$V_{\chi,j}^{u}=\begin{cases}\dfrac{\mathrm{X}_{j+1}-\mathrm{X}_{j}}{2} & j=1,2,\ldots,n_{\chi}-1\\ +\infty & j=n_{\chi}\end{cases}\tag{21}$$

$$V_{\chi,j}^{l}=\begin{cases}-\infty & i=1\\ \dfrac{\mathrm{X}_{j}-\mathrm{X}_{j-1}}{2} & j=2,3,\ldots,n_{\chi}-1\end{cases}.\tag{22}$$

The use of $+\infty$ for the highest bin upper cutoff and $-\infty$ for the lowest bin lower cutoff truncates the transition probability distributions, so that the probability of moving above the highest cutoff or below the lowest cutoff is accounted for in the probability of transitioning to the highest or lowest state, respectively.

With these quantities defined and the parameter values estimated for the TF model, we can calculate the probability of transitioning from a state $\left(\Xi_i,\mathrm{X}_j\right)$ to any state $\left(\Xi_k,\mathrm{X}_h\right)$. The transition probabilities are

$$\begin{aligned}&P\left(\xi_t=\Xi_k,\chi_t=\mathrm{X}_h \mid \xi_{t-1}=\Xi_i,\chi_{t-1}=\mathrm{X}_j\right)\\ &\qquad=P\left(V_{\xi,k}^{l}<\xi_t<V_{\xi,k}^{u},V_{\chi,h}^{l}<\chi_t<V_{\chi,h}^{u} \mid \xi_{t-1}=\Xi_i,\chi_{t-1}=\mathrm{X}_j\right)\end{aligned}\tag{23}$$

The increments of the two processes are correlated with the short-term factor dependent on the long-term factor. Hence we can rewrite Equation (23) as

$$\begin{aligned}&P\left(\xi_t=\Xi_k,\chi_t=\mathrm{X}_h \mid \xi_{t-1}=\Xi_i,\chi_{t-1}=\mathrm{X}_j\right)\\ &\quad=P\left(V_{\chi,h}^{l}<\chi_t<V_{\chi,h}^{u} \mid \xi_{t-1}=\Xi_i,\chi_{t-1}=\mathrm{X}_j\right)P\left(V_{\xi,k}^{l}<\xi_t<V_{\xi,k}^{u} \mid \xi_{t-1}=\Xi_i\right)\end{aligned}\tag{24}$$

Before moving on we introduce some new notation to facilitate use of the two-dimensional grid. Define the combined state space Σ for the two-dimensional lattice as $\Sigma_{(i-1)n_{\chi}+j}=\left(\Xi_i,\mathrm{X}_j\right)$, for $i=1,2,\ldots,n_{\xi}$ and $j=1,2,\ldots,n_{\chi}$. This indexing scheme matches the first n_{χ} states to $\left(\Xi_1,\mathrm{X}_j\right)$ for $j=1,2,\ldots,n_{\chi}$, the second n_{χ} to $\left(\Xi_2,\mathrm{X}_j\right)$ for $j=1,2,\ldots,n_{\chi}$, and so forth.

χ_t is normally distributed with mean μ_χ^c, conditioned on ξ_t transitioning from state Ξ_i to a state Ξ_k, and standard deviation σ_χ^c, conditioned on the standard deviation of ξ_t, σ_ξ. The conditional standard deviation of χ_t is given by

$$\sigma_\chi^c = \sqrt{\Delta\sigma_\chi^2 - \Delta\sigma_\xi^2 a^2}\,, \tag{25}$$

where a is a regularization coefficient given by

$$a = \rho \frac{\Delta\sigma_\chi}{\Delta\sigma_\xi}\,. \tag{26}$$

The quantities $\Delta\sigma_\chi$ and $\Delta\sigma_\xi$ are the standard deviations of increments of χ_t and ξ_t, respectively, over time increments of Δt. Equations (27) and (28) give their respective values.

$$\Delta\sigma_\chi = \sigma_\chi \sqrt{\frac{1-e^{-2\kappa\Delta t}}{2\kappa}} \tag{27}$$

$$\Delta\sigma_\xi = \sigma_\xi \sqrt{\Delta t} \tag{28}$$

σ_χ and σ_ξ are the standard deviations of the two processes, and κ the mean reversion coefficient, estimated from futures data. Equation (29) gives the conditional mean of χ_t conditioned on transitioning from state $\Sigma_{(i-1)n_\chi+j} = \left(\Xi_i, \mathrm{X}_j\right)$ to a state with long-term factor Ξ_k. $\overline{x}$ is the unconditional mean of χ_t.

$$\mu_{\chi,(i-1)n_\chi+j,k}^c = \overline{x} + \left(V_{\chi,j} - \overline{x}\right)e^{-\kappa\Delta t} + a\left(V_{\xi,k} - V_{\xi,i}\right) \tag{29}$$

Equations (27) and (23) are results from the mean and variance of the correlated bivariate normal distribution of (X, Y) conditioned on Y. Now we can compute the probability of moving from state $f = \left(i-1\right)n_\chi + j$ to state $g = \left(k-1\right)n_\chi + h$ for each entry in the $|\Sigma| \times |\Sigma|$ transition probability matrix T using equations (30) – (34).

$$p_{u,f,g} = P\left(N\left(\mu_{\chi,f,\left\lfloor \frac{g-1}{n_\chi} \right\rfloor+1}^c, \sigma_\chi^c \right) \le V_{\chi,\mathrm{mod}\left(g-1,n_\chi\right)+1}^u \right) \tag{30}$$

$$p_{l,f,g} = P\left(N\left(\mu_{\chi,f,\left\lfloor \frac{g-1}{n_\chi} \right\rfloor+1}^c, \sigma_\chi^c \right) \le V_{\chi,\mathrm{mod}\left(g-1,n_\chi\right)+1}^l \right) \tag{31}$$

$$q_{u,f,g} = P\left(N\left(\Xi_{\left\lfloor \frac{f-1}{n_\chi} \right\rfloor +1} + \mu_\xi \Delta t, \sigma_\xi \right) \le V^u_{\xi, \left\lfloor \frac{g-1}{n_\chi} \right\rfloor +1} \right) \tag{32}$$

$$q_{l,f,g} = P\left(N\left(\Xi_{\left\lfloor \frac{f-1}{n_\chi} \right\rfloor +1} + \mu_\xi \Delta t, \sigma_\xi \right) \le V^l_{\xi, \left\lfloor \frac{g-1}{n_\chi} \right\rfloor +1} \right) \tag{33}$$

$$T_{f,g} = \left(p_{u,f,g} - p_{l,f,g} \right)\left(q_{u,f,g} - q_{l,f,g} \right) \tag{34}$$

$N\left(\mu,\sigma\right)$ denotes a normal random variable with mean μ and standard deviation σ .

3.3.2. Grid Parameters

In this section we provide the parameters we use for the grid approximation and discuss their justification. Table 3 lists the values for each of the lattice parameters. Rather than discretizing the two processes of TF into an equal number of states, we use 150 states for the long-term factor, and 30 for the short-term factor. The reasons for these choices are explained below. As shown in §3.1.5, the variance of the long-term factor increases as the time period over which the process varies increases, while the variance for the short-term factor quickly approaches a steady limit. More states are needed to adequately capture the wide range of possible values of the long-term factor.

Parameter	Symbol	Value
Number of deviation states	n_χ	30
Number of equilibrium states	n_ξ	150
Equilibrium lower cutoff value	v_ξ^l	2
Equilibrium upper cutoff value	v_ξ^u	6.5
Deviation lower cutoff value	v_χ^l	-1.3
Equilibrium upper cutoff value	v_χ^u	1.3

Table 3: Grid approximation parameter values.

Figures 10, 11, and 12 show the values given by GBM, MR, and TF, respectively, as a percentage of the "true value" of a 40 year stream of unit production in each time period, over a range of the number of states in the grid for each. The true values, determined from the analytic mean equations given in §3.1, provide a benchmark we use to find good approximations. Observation of the valuations for each model as the number of states increased provides a guide for choosing the specific parameters. In Figure 10 we see that 150 equilibrium (long-term factor) states for GBM approximate the benchmark value to within 2%. Similarly, in Figure 11 we see that 30 deviation (short-term factor) states for MR also approximate its benchmark to within 2%. These numbers of equilibrium and deviation states also very closely approximate the benchmark for TF to within 0.5%.

The lower and upper equilibrium factor cutoffs correspond to prices of \$7.39 and \$665.14, respectively. Including the short-term factor deviations gives a full range of prices of \$2.01 to \$2,440.60. The probability of the price falling outside of this very wide range is negligible. The short-term factor range encompasses the short-term factor distribution from the 0.5^{th} percentile to approximately the 100^{th} percentile.

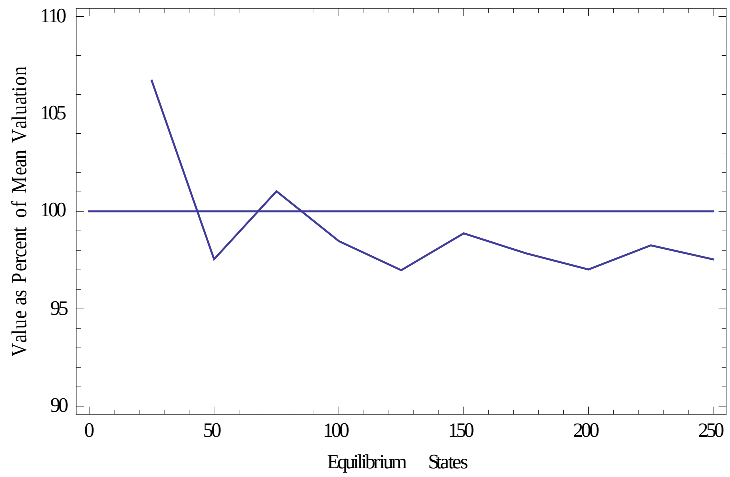

Figure 10: Grid convergence for GBM.

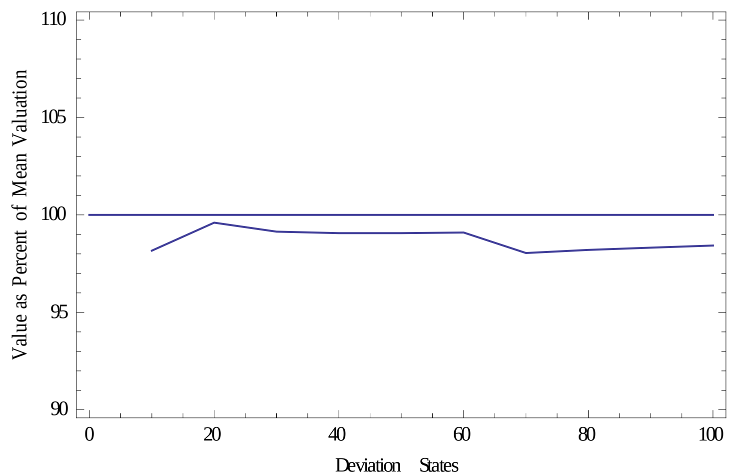

Figure 11: Grid convergence for MR.

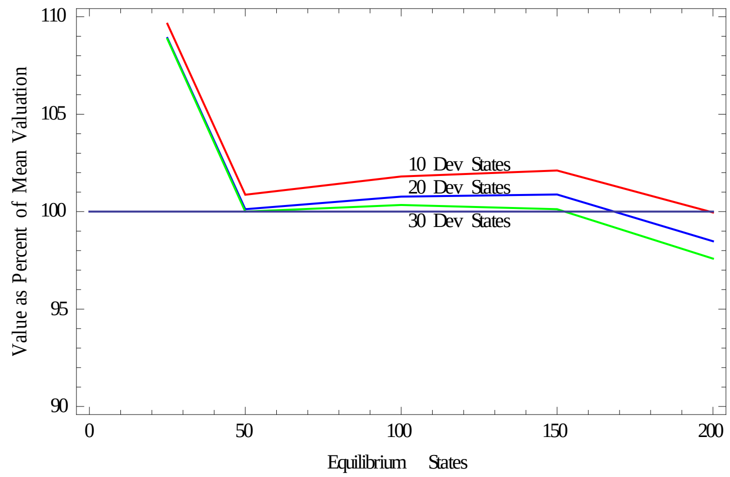

Figure 12: Grid convergence for TF.

4. Portfolios of Oil Projects

4.1. Project Model

We use a production model that describes the rate of oil production over time for an oil field development project. This production model fully describes the projects we use to build portfolios for our analysis. From the decision-analytic perspective, we are looking at a portion of the larger decision tree that includes all the decisions from exploration and appraisal to development and extraction. The fields have already been appraised and we now face the decision of which projects to invest in. All costs before this point are sunk costs, and as such do not factor into the analysis. To look at the average behavior of the price models over many different possible scenarios, we use Monte Carlo simulation to generate candidate sets of projects from which to build portfolios. To achieve this, we assign distributions to the parameters of the production model, which are described below. Note that we use Monte Carlo simulation to generate *sets of projects*, not to simulate prices, which are instead handled in the grid approximation described above.

We use the production model in Al-Harthy (2007) as the basis for our model. This model has several parameters that define an oil production profile over time. The *Start Year* is the development time between initialization of the project and the time production begins. Once production begins, it starts at an amount equal to the *Peak Production* parameter, and produces at this level for a time specified by the *Plateau Length*. Production is assumed to be limited to *Peak Production* by technological constraints, and once a certain amount of the reservoir is depleted, the rate falls below this level and declines at the rate *Decline Rate*. Production ends when the production rate falls below the *Economic Field Limit*. The project incurs an *Annual Operating Cost* and a per-barrel *Unit Production Cost*, as well as an up-front capital expenditure, *CAPEX*. An example production profile using the parameters from Al-Harthy (2007) appears in Figure 13, and the parameters themselves appear in Table 4.

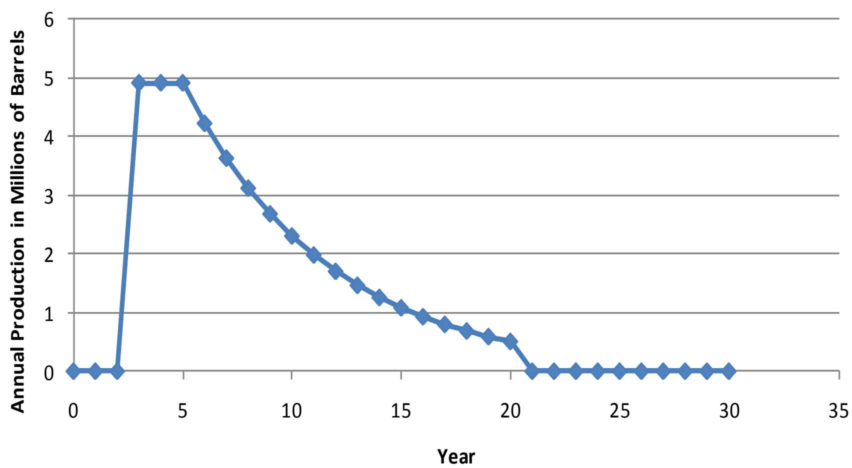

Figure 13: Annual Production for Al-Harthy's Project Model.

Parameter	Units	Value
Start Year	Years	3
Plateau Length	Years	2
Peak Production	MMBL/Year	4.92
Decline Rate	%/Year	14
CAPEX	$MM	300
Economic Limit	MMBBL	0.39
Annual Operating Cost	$MM	15
Unit Production Cost	$	2

Table 4: Al-Harthy (2007) production model parameters.

We generate sets of projects having identical parameter distributions using Monte Carlo simulation, and build production profiles according to the generated values for the parameters described below. The *Start Year*, *Peak Production*, and *Plateau Length* are taken as having uniform distributions and the *Decline Rate* and *CAPEX* as lognormal. The *Economic Field Limit*, *Annual Operating Cost*, and per-barrel *Unit Production Cost* are taken as fixed. The parameters for the variable project parameters are given in Table 5 and the values of the fixed project parameters are given in Table 6.

In reality a variety of technical and geological factors determine each of the production parameters. The parameters we use are based on the parameters in Al-Harthy (2007), but adjusted for differences in the mean oil price, which is explained below. We make minimal assumptions about the parameter distributions to make the model as general as possible. The only assumptions we make about *Start Year*, *Plateau Length*, and *Peak Production* are that they are bounded over a range based on Al-Harthy's (2007) parameters. *Start Year* and *Peak Production* are bounded to within 50% above or below Al-Harthy's (2007) parameters and *Plateau Length* is bounded between zero and Al-Harthy's (2007) parameter. *Start Year* and *Peak Production* must each be some positive number since there is some amount of setup time and some amount of initial production, but *Plateau Length* can be near zero depending on whether the technological constraints are binding on the initial production. Based on these minimal assumptions, we draw values for these three parameters from uniform distributions over their respective ranges.

Similarly for *CAPEX* and *Decline Rate*, we restrict our assumptions to those of the parameters being positive and unbounded, with values near the mean more likely. For parameters we use a lognormal distribution having the corresponding parameter from Al-Harthy (2007) as the mean, and a standard deviation empirically determined so that the distribution of the projects' internal rates of return (IRR) has significant variation but is still reasonable.

Parameter	Units	Distribution	a	b
Start Year	Years	Uniform(a,b)	Min = 2	Max = 4
Plateau Length	Years	Uniform(a,b)	Min = 0	Max = 2
Peak Production	MMBL/Year	Uniform(a,b)	Min = 3.6	Max = 7.2
Decline Rate	%/Year	LogNormal(a,b)	$\mu = 0.15$	$\sigma = 0.01$
CAPEX	$MM	LogNormal(a,b)	$\mu = 1200$	$\sigma = 400$

Table 5: Parameters for variable project characteristics.

Parameter	Units	Value
Economic Limit	MMBBL	0.07
Annual Operating Cost	$MM	45
Unit Production Cost	$	6

Table 6: Parameters for fixed project characteristics.

We adjust the *CAPEX*, *Annual Operating Cost*, and *Unit Production Cost* parameters from Al-Harthy (2007) so that project economics, particularly IRR, are reasonable. Al-Harthy uses an initial spot price of $25, compared to our initial spot price of approximately $75, so we accordingly increase these parameters by a factor of three. The costs associated with production tend to be correlated with oil price as providers of certain production services or equipment can charge higher prices when oil price increases. We ignore cost uncertainty, which might correspond to the situation of locking in equipment or service costs through a contract with the supplier at the start of the project. Figure 14 shows the distribution of project IRRs over 5,000 projects generated according to the parameter distributions given above in Table 5 and Table 6. The project IRRs have a sample mean of 11.13% and standard deviation of 7.76%.

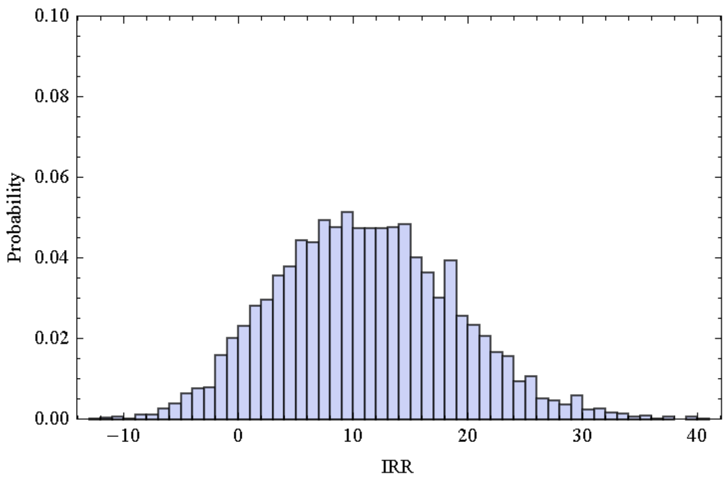

Figure 14: Histogram of the IRRs of 5,000 generated projects.

Later in our analysis we consider options embedded within projects that have this production model. We focus primarily on an option described in Brandão, Dyer, and Hahn (2005) to either buyout a partner's share of the project, divest our share, or do nothing, at a single exercise date. We refer to this as the buyout/divest option. Smith (2005) calls this kind of option a "scale" option, since it scales the magnitude of future cash flows of the project.

4.2. Valuing Projects and Options Using the Grid Approximation

We use the grid to value both the projects themselves without any options and the scale options we consider. The value of a scale option for a project is a function only of the project value at the time period under consideration. In other words, the value of an option held today, which can be exercised today, on a project that extends some time into the future is related to the project only through the present expected value of the project. This implies that the option can be valued separately from the project, and that the values are additive; the value of the project with the option is the sum of the values of the project and option individually. We can determine the value of a project at every price state in every time period (which also gives us the present value of the project at every initial price state) and use these values to determine the value of a scale option at every price state for every time period. We first derive the project valuation procedure, then option valuation for European and American "put" and "call" options, and finally the buyout/divest option on the project.

A project with no options can be succinctly described as a series of cash flows. The oil projects we consider have an initial negative cash flow at time $t = 0$ for the capital expenditure, *CAPEX*, then no other cash flows until production begins. The cash flows associated with production depend on oil price, as they are the market values of the oil produced in each period Δt, less the costs incurred in that period. The costs and production rates are deterministic and independent of oil price, so the cash flow of a period is fully determined for a given oil price.

Valuing the cash flows that make up the project involves working backward from the time of the last nonzero cash flow, the project termination date, τ_p. The project termination date is defined as the last time period before the production level falls below the economic field limit, at which time production is stopped. With this "backward recursion" we get the value of the project at each price state in each time period, which we need for option valuation (Luenberger 1998). For each price state we add the single-period cash flow to the discounted expected value of the project in the next period. We denote the project value at time *t* in price state S_t by $V_p(S_t, t)$. At period *t*, for a project with production P_t, fixed cost c_f, and unit cost c_u, for an oil price S_t the net cash flow in that period, C_t is given by

$$C_t = P_t S_t - P_t c_u - c_f. \tag{35}$$

For each time period before τ_p, we add the expected value of the future cash flows, discounted over Δt at rate *r*, which is a recursion given by

$$V_p\left(S_{\tau_p}, t\right) = C_t + e^{-r\Delta t}\mathrm{E}\left[V_p\left(S_t, t+1\right)\right] \quad t < \tau_p. \tag{36}$$

The expectation in Equation (36) is taken over the conditional transition probability distribution, defined by the i^{th} row in the transition matrix *T*. Writing the expectation explicitly in terms of project values and transition probabilities given that the price process is in state *i* at time *t* gives

$$\mathrm{E}\left[V_p\left(S_t, t+1\right)\right] = \sum_{j=1}^{|\Sigma|} T_{i,j} V_p\left(S_j, t+1\right). \tag{37}$$

At the termination time τ_p there are no future cash flows in the project, so the value of the project at that time is simply the cash flows of that period, C_{τ_p}:

$$V_p\left(S_{\tau_p}, \tau_p\right) = C_{\tau_p} \tag{38}$$

The equations needed to value the project at each price state in each time period up to time τ_p are then

$$V_p\left(S_{\tau_p}, \tau_p\right) = C_{\tau_p} \tag{39}$$

$$V_p\left(S_{\tau_p}, t\right) = C_t + e^{-r\Delta t}\sum_{j=1}^{|\Sigma|} T_{i,j} V_p\left(S_j, t+1\right) \quad t < \tau_p. \tag{40}$$

We now describe valuation of a project with an option. As already mentioned, the option and project can be valued separately. This requires that the value of the project has already been determined at each price state and each time period when exercise is possible. We first describe valuation of a European-style option, which has a single exercise date, and then an American-style option, which can be exercised at any time up to expiration of the option.

Similar to the project valuation procedure, for the option, the value is first determined at expiration for each price state, and then the values at each previous time step are determined by working backward from expiration to the present. The procedure is defined by a recursive relation where the option value at each time step is a function of the discounted expected option value at the subsequent time step. At any time period that option exercise is possible, for each price state, we need to determine if it is optimal to exercise the option.

The two basic kinds of options are "calls" and "puts". A "call" option gives the option holder the right, but not the obligation to buy the underlying asset at a predetermined price. A "put" option gives the holder the right, but not the obligation to sell the asset at a predetermined price. The buyout/divest option is a combination of a call and a put, since it allows the holder to either buy an asset *or* sell an asset, each at a predetermined price.

At expiration for the European-style option, the only time the option can be exercised, the value of the option is the greater of either exercising or letting the option expire. Exercising a call option to buy out a partner's share, or otherwise immediately expand production, involves paying a predetermined price K to increase future cash flows of the project by a multiplicative amount f. If the option is not exercised, then it expires worthless. Then we can calculate the value V_o of the option at expiration time τ_o for each price state S_{τ_o} by

$$V_o\left(S_{\tau_o},\tau_o\right)=\max\left\{fV_p\left(S_{\tau_o},\tau_o\right)-K,0\right\}. \tag{41}$$

To find the present expected value of the option, we work backward one time period at a time from the expiration date taking the expectation of the option value at each price state, discounting it one time period. The recursion describing this procedure is given by

$$V_o\left(S_t, t\right) = e^{-r\Delta t}\mathrm{E}\left[V_o\left(S_t, t+1\right)\right] \quad t < \tau \tag{42}$$

where the expectation is the sum of the values of the options in each price state times the probability of transitioning to that state:

$$\mathrm{E}\left[V_o\left(S_t, t+1\right)\right] = \sum_{j=1}^{|\Sigma|} T_{i,j} V_o\left(S_j, t+1\right) \tag{43}$$

The equations for finding the present value of this option are then

$$V_o\left(S_{\tau_o}, \tau_o\right) = \max\left\{ fV_p\left(S_{\tau_o}, \tau_o\right) - K, 0\right\} \tag{44}$$

$$V_o\left(S_t, t\right) = e^{-r\Delta t}\sum_{j=1}^{|\Sigma|} T_{i,j} V_o\left(S_j, t+1\right) \quad t < \tau\,. \tag{45}$$

For a put option, the value at expiration is instead the greater of either selling the asset at a predetermined price *D*, or continuing to hold the asset, the latter of which has a value of the expected value of the asset at that price state and time. The asset here is the project under consideration, which has a value $V_p\ S_{\tau_o}, \tau_o$. Then for a put option, Equation (44) becomes

$$V_o\left(S_{\tau_o}, \tau_o\right) = \max\left\{D, V_p\left(S_{\tau_o}, \tau_o\right)\right\}, \tag{46}$$

and Equation (45) remains the same.

Another option we consider is an American-style option, which is similar to the European option, with the exception that the option can be exercised at any time before expiration. This option gives the decision maker more flexibility. The equation for the value at expiration for an American option is identical to that for the European option. The recursive equation, however, incorporates the decision at each time period of whether to exercise or not. The equations for the American-style option are

$$V_o\left(S_{\tau_o},\tau_o\right)=\max\left\{fV_p\left(S_{\tau_o},\tau_o\right)-K,0\right\} \quad (47)$$

$$V_o\left(S_t,t\right)=\max\left\{fV_p\left(S_t,t\right)-K,e^{-r\Delta t}\sum_{j=1}^{|\Sigma|}T_{i,j}V_o\left(S_j,t+1\right)\right\} \quad t<\tau_o. \quad (48)$$

Similarly, for an American put option, Equations (47) and (48) become

$$V_o\left(S_{\tau_o},\tau_o\right)=\max\left\{D,V_p\left(S_{\tau_o},\tau_o\right)\right\} \quad (49)$$

$$V_o\left(S_t,t\right)=\max\left\{D,e^{-r\Delta t}\sum_{j=1}^{|\Sigma|}T_{i,j}V_o\left(S_j,t+1\right)\right\} \quad t<\tau_o. \quad (50)$$

Extension to the buyout/divest option is straightforward. Since this option gives the alternatives of buying out the partner to increase our share, completely divesting our share, or doing nothing, the value function for the option is simply the maximum of these three alternatives. For the European option we modify Equation (44) to include the divest alternative, which results in Equation (51). The first term in the $\max\{\}$ function in Equation (51) corresponds to buying out the partner, the second to divesting our share, and the third to doing nothing. Equation (45) does not need modification, since it does not involve any decision.

$$V_o\left(S_{\tau_o},\tau_o\right)=\max\left\{fV_p\left(S_{\tau_o},\tau_o\right)-K,D,0\right\}. \quad (51)$$

The American buyout-divest option requires modification of both Equations (47) and (48), since each includes the decision to exercise or not. The result is Equations (52) and (53).

$$V_o\left(S_{\tau_o},\tau_o\right)=\max\left\{fV_p\left(S_{\tau_o},\tau_o\right)-K,D,0\right\} \quad (52)$$

$$V_o\left(S_t,t\right)=\max\left\{fV_p\left(S_t,t\right)-K,D,e^{-r\Delta t}\sum_{j=1}^{|\Sigma|}T_{i,j}V_o\left(S_j,t+1\right)\right\} \quad t<\tau_o. \quad (53)$$

4.3. Measuring Differences Between Portfolios

Stochastic price models impact decision making if their use could potentially lead to a different decision than treating price deterministically. This, of course, assumes that the stochastic price model is reasonably constructed and more accurate than a fixed price. In this section we define the metrics we use to quantify any impact stochastic price models may have. We examine two general aspects of the portfolios constructed from the candidate project sets: comparison of portfolio composition, and comparison of portfolio value.

First we describe the analysis of a sample candidate project set to motivate the metrics. Suppose we have a candidate set of 20 projects, from which we choose a portfolio of 10. Suppose we have two price models with which we can value the projects. Two portfolios are constructed by choosing projects according to the valuations given by each price model. The projects are denoted by letters and the two portfolios by P1 and P2. Figure 15 shows the composition of the two portfolios, P1 and P2, chosen from the same candidate set of projects using the two different price models. Portfolio P1 includes projects A, B, C, D, F, and G, and portfolio P2 includes all of those in addition to projects E and H.

We are interested in measuring the difference between the two portfolios to determine how the portfolio composition decision changes with the price model used. The order of the projects in Figure 15 represents their rankings. Both price models rank project A first, then project B, etc. The portfolios at the top of the figure simply show the rankings, which we refer to as "rank portfolios", or as "*k*-rank portfolios" for portfolios composed of the top *k* ranked projects (here $k = 10$). In addition to rank, each project has a value, which may be negative. The portfolios at the bottom of the figure indicate with grey ovals which projects are excluded for having negative value. We refer to these as "rank portfolios with exclusion."

Ranks for k=10:

P1:	A	B	C	D	G	F	H	E	I	K
P2:	A	B	C	D	E	F	G	H	I	J

+NPV portfolios for k=10:

P1:	A	B	C	D	G	F	H	E	I	K
P2:	A	B	C	D	E	F	G	H	I	J

The grey ovals indicate projects not included in the portfolio

Figure 15: Two portfolios P1 and P2 with k = 10 and projects denoted by letters.

Notice that the rankings are different for some of the projects. A simple measure of portfolio difference is the fraction of projects both models ranks the same, a metric we call Rank Agreement (RA). In this example the RA is 0.6; out of the 10 projects, 6 have the same rank. This measure is strict and myopic, since difference in rankings does not preclude identical portfolios. However, we can say that if the models agree on a high fraction of project ranks, then they are likely to produce similar portfolios.

Looking at portfolios directly, we obtain a better measure of difference by determining the fraction of k projects that both models include in a k-rank portfolio, which we call the k-Portfolio Agreement (kPA). For our example this fraction is 0.9, meaning that 90% of the projects in the two portfolios are included in both.

In Figure 16 the projects in each portfolio have been rearranged to show where they overlap and where they differ. The rankings for each project are included as a reminder that the orders in the portfolio are no longer necessarily the rank orders. The elements with circles denote empty "slots." The price model used for portfolio P1 assessed only six projects with positive value, while the price model for P2 assessed eight as positive. The “common projects” are those that both models valued as positive, and included in both *k*-rank portfolios. “Project differences” are those projects that are either in only one rank portfolio or in both but valued positively by only one price model. “Common empty projects” represents the available portfolio “slots” left unfilled by both price models. This last case only occurs when both price models value fewer than *k* projects as positive.

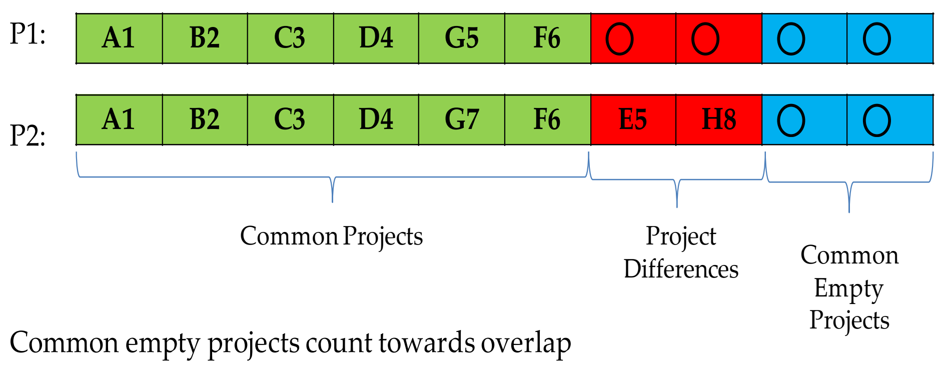

Figure 16: The two portfolios rearranged to highlight differences.

Using Figure 16 as a guide, we can measure the similarity of the two portfolios constructed using project values rather than ranks alone. This measure, *k*-Portfolio Agreement with Exclusion (kPAE), is the most realistic of those described thus far, as it allows the exclusion of unattractive projects. Both the Common Projects and the Common Empty Projects count towards the percentage of agreement. The two valuation methods agree on 80% of a 10 project portfolio for the given candidate project set.

The second analysis is to compare the values of portfolios constructed using different price models. We use each model to select which projects to include in a *k* project portfolio, but determine the actual value using TF. This gives the best approximation to the "true" portfolio values, which when compared to the value of the portfolio constructed using the TF model yields the value lost by using a model other than TF. This measure, which we refer to as "Value Lost", quantifies the bottom-line impact of any differences in decisions. It, in conjunction with the portfolio similarity measures described above, describes the degree of differences in decision making and the impact of those differences.

4.4 Monte Carlo Simulation

We are interested in the average effects of using different price models over many possible situations. We use Monte Carlo simulation to generate different candidate project sets using the project parameter distributions described in §4.1. For each generated candidate project set, we compute the metrics described in §4.3. The simulation and all calculations are performed in Mathematica 8.0.

For each of the 500 randomly generated candidate sets, we calculate the metrics for each portfolio size from $k = 1$, choosing the best project, to $k = 20$, the entire candidate set. The averaged metrics for each k over all Monte Carlo iterations provide the results for our analysis.

5. Results and Discussion

In this chapter we present several results which support our conclusions. The chapter is broken into two parts to delineate the cases in which projects have no options from those in which they do. Arguments for the use of stochastic price models in valuing options motivate this distinction, since projects with options can only be valued correctly if price is modeled stochastically.

5.1. No-Option

Here we consider building portfolios when the projects do not have options. First we compare rankings by different models and portfolios constructed using those rankings when project values are not considered. Then we consider portfolio construction with exclusion of negative value projects. In these cases, we find F to be a viable substitute for TF, but as we show the price used for F cannot be arbitrary. We then compare the three-point discretization to TF, and see that a slight increase in complexity leads to a distinct increase over F in ranking agreements.

5.1.1. Rankings

First we consider rankings alone without valuation using the RA and kPA metrics described in the previous chapter. Table 7 shows the RA for each model compared to TF. F has the lowest agreement, with a RA of 68.3%, however this means that two-thirds of the projects are ranked the same, on average, by the simplest and most complex models we consider. GBM has the highest RA at 95.9%, suggesting that the short-term factor of TF is of little consequence to project rankings, which makes economic sense. The RA measure is rather strict, but already suggests that price models may not have much impact on decision making in the absence of options.

Model	Percent of projects ranked same as TF
GBM	95.9%
MR	78.2%
F	68.3%
FS	85.5%

Table 7: Average RA for projects without options.

We look now at the kPA for each portfolio size *k* from 1 to 20, which is shown in Figure 17. This shows the agreement between portfolios constructed by selecting the top *k* ranked projects according to each model. Along the horizontal axis are the different values for *k*, and along the vertical axis are the average fraction of projects in each models' *k*-portfolio that is also included in the TF *k*-portfolio - the kPA. The left figure shows the full scale of kPA from 0 to 1; it is clear from the figure that all the agreements are close to 1. The figure on the right gives a detailed view of the kPA scale from 0.9 to 1.

The highest disagreements, although quite small, occur over the smaller portfolios, as one would expect since relative values play a larger role in this case. MR, F, and FS each give the same best project as TF about 94% of the time, while GBM agrees with TF on the best project over 99% of the time. Considering a portfolio of the top two projects, the agreement with TF is 99.6% for GBM, 95.6% for MR, 96.5% for F, and 94.5% for FS. These are the average fraction of projects that are the same between portfolios created using different models, but for a given portfolio this fraction can only be 0%, 50%, or 100%. This implies that the majority of the time two models agree on both projects; only occasionally disagreeing on one or both. Portfolios of 5 projects or more, on average, agree with TF on over 95% of the portfolios. A portfolio of 20 projects is clearly identical for all the models since each portfolio contains the entire set of candidate projects.

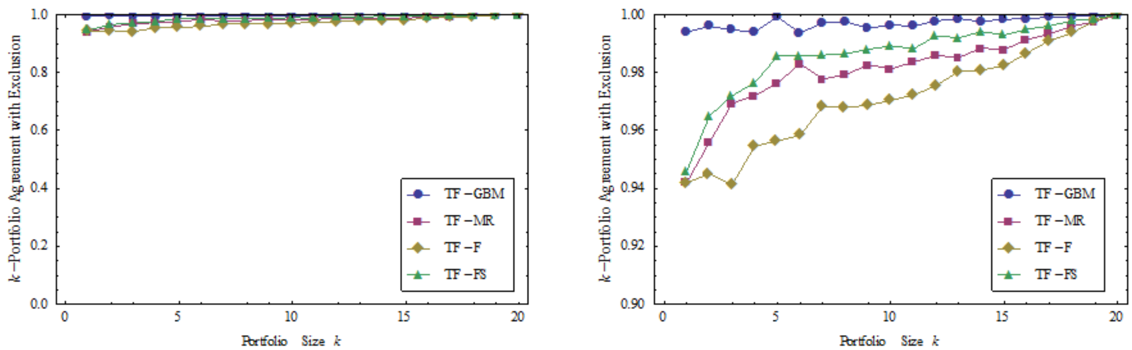

Figure 17: Average kPA for projects without options.

While there are some differences in portfolio composition between the different models, even the best and worst models agree highly. If only rankings are considered, a fixed price can be used for valuation with very little impact on decision making. In the next section we consider valuations in addition to rankings.

5.1.2. Rankings with Exclusion

The stochastic models clearly have little impact when considering rank alone. These results are useful as long as the candidate projects all have positive values, so now we consider the situation where some projects have negative values and are excluded from portfolios. Figure 18 shows the kPAE for each portfolio of size k. For $k \leq 8$, the kPAE is approximately the same as the kPA, but for $k > 8$ the kPAE differs from the kPA, distinctly so for F and less significantly for MR. Agreement between F and TF falls at its lowest to 83.7%. MR agreement with TF falls to 92.0% at its lowest.

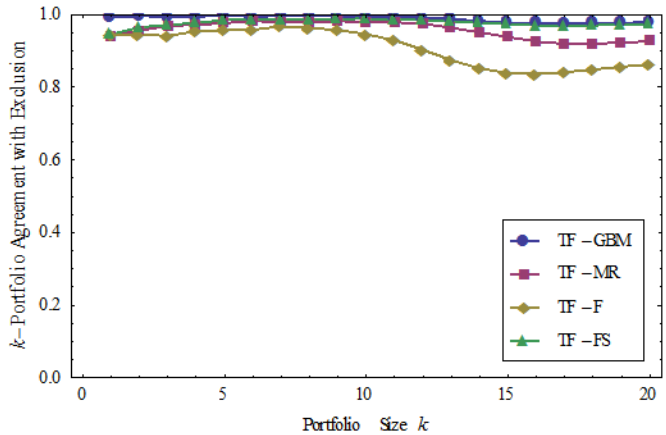

Figure 18: Average kPAE for projects without options.

Excluding negative projects from the portfolios shows more noticeable differences between the models' portfolios, because each model values a particular project differently. We consider the impacts of these valuation differences below.

If a model undervalues (overvalues) projects, it will generally include fewer (more) projects than the ideal TF portfolio. A model that overvalues projects, for example, may value some projects as positive that have negative values according to our gold standard, TF. Use of this overvaluing model can lead to decisions to include more projects than is optimal in large enough portfolios (k is greater than the number of positive-value projects).

Table 8 shows the average number of positively valued projects and the average project value for each model. Notice that the kPAE becomes lower than the kPA when k is near or above the average number of positively values projects. When k is lower than the average number of positive valued projects for both the model in question and TF, it is very likely that the k-portfolio can be filled with positively valued projects, and thus exclusion will have little impact on the portfolio composition decision.

Model	Avg. Number of Positive Valued Projects	Avg. Project Value ($MM)
TF	13.96	363.48
GBM	13.53	323.43
MR	15.43	541.44
F	11.23	100.87
FS	14.49	425.29

Table 8: Average number of positively valued projects and average project value for each model.

The kPAE is lower than the kPA for larger portfolios, meaning a different decision is more likely, but how does this affect the value associated with a decision? An undervaluing model leads to choosing fewer projects than is optimal, so value is passed up. An overvaluing model leads to choosing more projects than is optimal, so projects with negative values are included. In both cases the value is lower than the optimal portfolio value obtained using TF. The differences between these portfolio values and the TF portfolio value is the value lost by using a less descriptive model. For models and portfolio sizes where the kPAE is high, the value lost is low. Over all values of *k* between 1 and 20, GBM and FS have values lost of less than 0.2% of the optimal portfolio value. As the kPAE decreases for MR and F, the value lost for the two models increases. The maximum value lost for MR is 1.2%, and for F is 3.9%. The value lost for each *k* is shown in Figure 19.

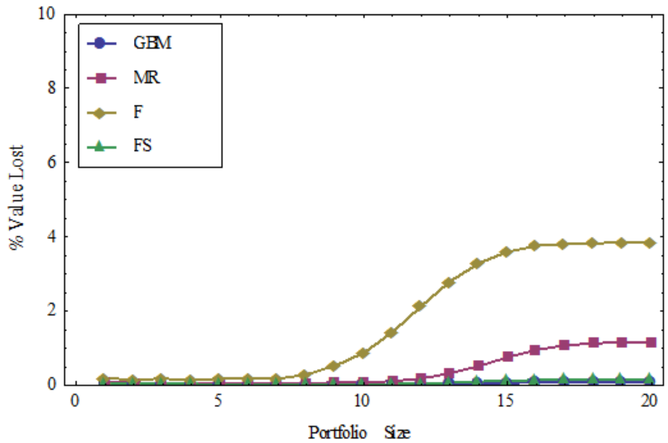

Figure 19: Value lost by using each model instead of TF.

We can see in Figure 19 that the value lost for F and MR level off around k = 17. Lost value is primarily an effect of differences in valuation. Figure 19 only shows the value lost for one set of project distribution parameters, but we might be interested in the maximum value lost if the parameters change. A simple way to uniformly change project values is to add a certain amount to each projects' CAPEX. This effectively increases or decreases the threshold at which each model values a project as having positive or negative value. Figure 20 shows the value lost in millions of dollars ($MM) for a k = 20 project portfolio for each model. Decreasing CAPEX by $1,150MM on the low end causes nearly all projects to be chosen by each model. Increasing CAPEX by $2,000MM on the high end causes nearly all projects to be rejected. A change of $0 corresponds to the k = 20 points in Figure 19. The scale in Figure 20 is dollar values rather than percentage of the TF portfolio value because when the TF portfolio is empty (thus having no value) and another portfolio is not, computing the percentage value lost involves division by zero.

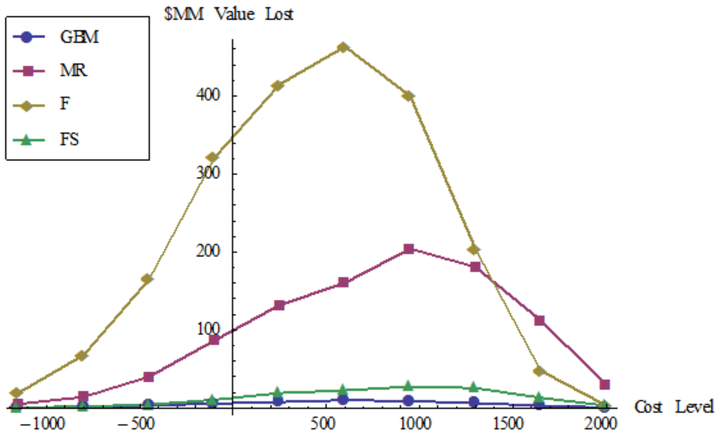

Figure 20: Maximum value lost for each model over a range of cost levels.

The impact of using a model other than TF is relatively small in most cases. Only when valuation becomes important for project exclusion does the value impact become non-negligible. In the case we consider, even when this impact in non-negligible it is a small percentage of portfolio value. These results indicate that modeling price as a single fixed value leads to good decisions when there are no options associated with the project. In the next section we consider the choice of what price to use for F.

5.1.3. Flat Price for Decision Making

Thus far we have seen that when projects do not have options, a simple fixed price, F, leads to nearly identical decisions as the more complicated TF. However, the price we use for F is close to the initial TF mean. In this section we consider the impacts of using different prices for F.

A common practice is to use a fixed price significantly lower than the average oil price, with the reasoning that if a project is profitable at an intentionally underestimated price, it will certainly be profitable at a higher price. Different fixed prices can lead to different project rankings, as can be seen in Figure 21, which shows the kPA for fixed prices of $50, $100, $150, and the fixed price used thus far: $82. The results of the previous sections hold for a fixed price close to the recent historical mean. Prices such as \$50 and \$150, which are significantly different from the mean, show distinct decreases in kPA for all values of *k*. The purpose of using an intentionally low price is to provide a margin of error and reduce the risk of loss, but this practice can lead both to incorrect valuations and incorrect ranks, passing up valuable projects due to underestimating their value. Using a fixed price F that is significantly lower or higher than long run averages leads to different project rankings, and decreases agreement between F and TF.

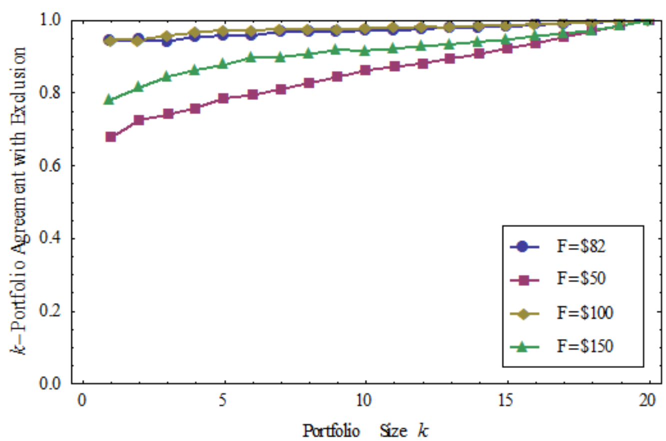

Figure 21: kPA for different fixed prices.

Figure 22 shows four sets of 20 projects with their rankings using fixed prices between \$10 and \$200 at \$10 increments. These graphs show examples of how project ranks directly change as the fixed price changes. Notice that some projects change drastically in rank over the range, but most do not move significantly.

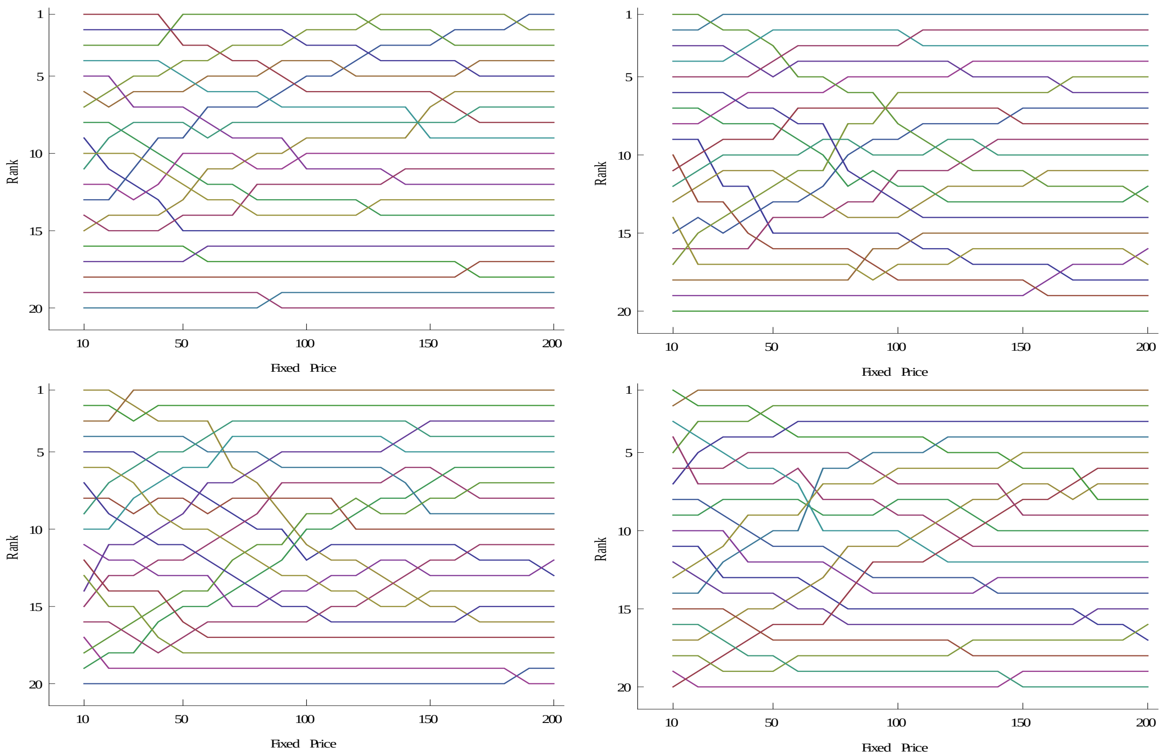

Figure 22: Rankings of 20 projects from four randomly generated candidate sets using fixed prices from \$10 to \$200.

Different projects have different levels of sensitivity to oil price. These sensitivities are the slopes of the lines in Figure 23, where each line corresponds to one project in a candidate set of 20. As the fixed price used for valuation increases, the values of different projects increase at slightly different rates. A high producing project with a significant CAPEX requirement can be more valuable at high prices than a lower producing, but less expensive project, that is more valuable at lower prices. In practice CAPEX is not independent of oil price, but we are considering a fixed CAPEX for each project and a range of oil prices used for valuation. The modeling choice of what fixed price to use for F can affect the rankings if this choice is far from the current mean price. Notice that although projects can have differing sensitivities, the slopes are not drastically different, and thus as in Figure 22 most of the ranks do not change much with different fixed prices.

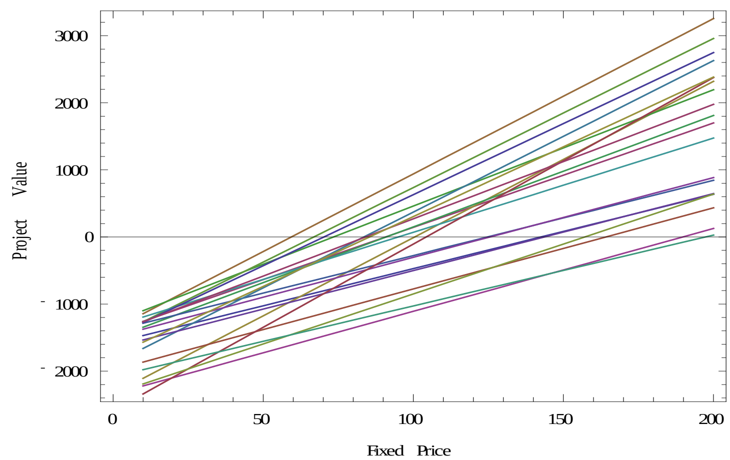

Figure 23: Project values of one set of candidate projects vs. the fixed price used for valuation.

5.1.4. P90, P50, P10 Discretization

Briefly we consider the discretization model (D) described in §3.2.1. Table 9 shows the average project valuation using D as compared to the other models. The average valuation for D is better than F, but farther from the TF value than each of the other models.

Model	Average Project Value ($MM)
TF	363.48
GBM	323.43
MR	541.44
F	100.87
FS	425.29
D	159.40

Table 9: Average project values for each model, including the discretization.

The kPA for D is also uniformly better than F for all values of *k*, as seen in Figure 24, although only by a margin of about 1%. This approach may be better suited to modeling other uncertainties that have a single, but unknown, value. While better describing uncertainty in oil price than F, D does not describe uncertainty over time. Its small improvement over F may be motivation for further investigation of its use.

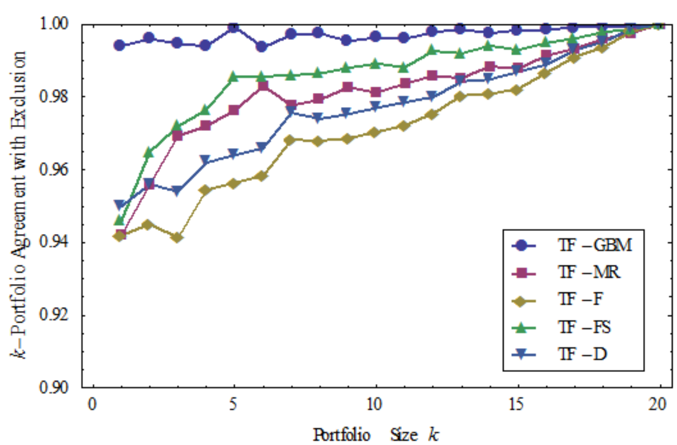

Figure 24: RA for each model including the P90, P50, P10 discretization.

5.1.5. Sensitivity to Initial Price

In this section we look at the sensitivity of these results to the initial price. The results would not be very useful if they were very sensitive to the initial price, and any change in price required a new parameter fit.

For FS, since it is derived directly from market data (it *is* market data), we assume that a price change is reflected in the forward and futures prices. To create a realistic price series, we first fit an exponential curve plus a drift factor to the current FS data to determine the reversion and drift factors, and then use these factors to create a new FS price series with a different starting price, but the same long term drift.

The earliest a project can start, according to the distribution of *Start Year* given in §4.1, is after 2 years from the beginning of development. Figure 25 shows the expected TF prices for several initial prices around \$74.27, ranging from \$30.11 to \$222.47. The initial price is changed by changing the initial short term state. Most of the deviation effects of a different initial price disappear due to mean reversion before year 2, denoted by the dashed line, or before any projects begin production. Figure 26 shows the results of changing the initial short term factor for MR, which are almost identical to TF. The price series for different initial prices for FS are shown in Figure 27. As expected, the behavior is similar to TF and MR, although the reversion rate is slightly slower.

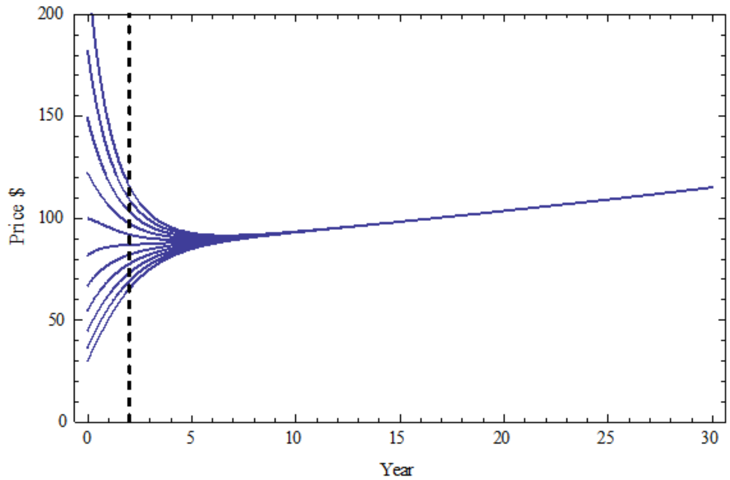

Figure 25: Expected TF prices for a variety of initial prices.

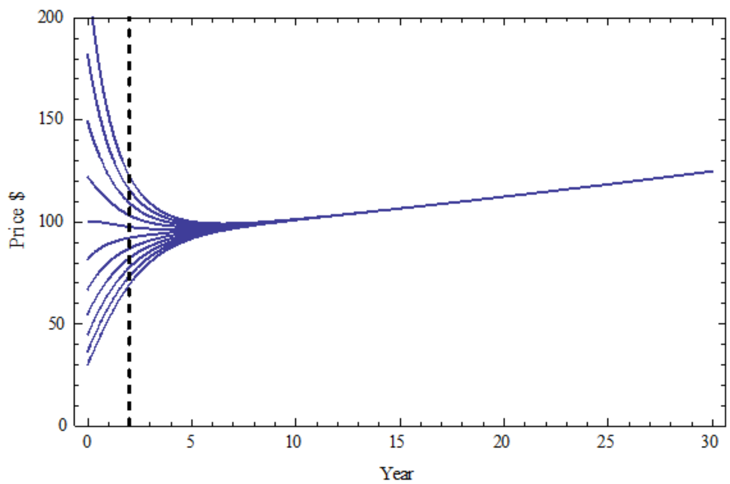

Figure 26: Expected MR prices for a variety of initial prices.

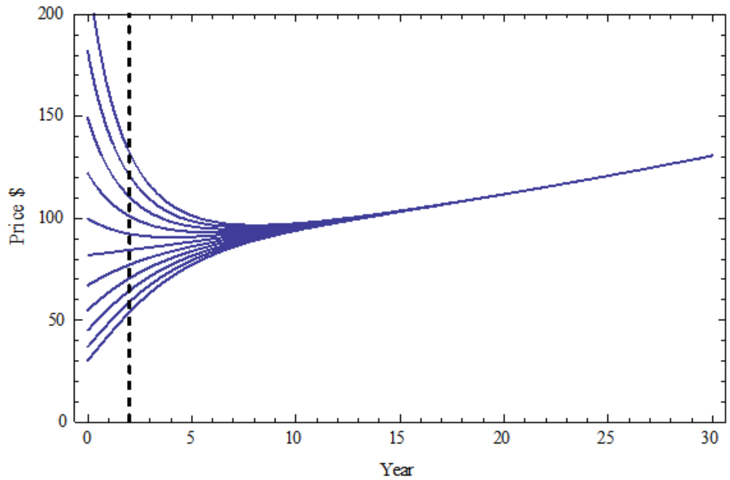

Figure 27: FS prices using a fitted model for a variety of initial prices.

Looking at Figure 28 we see that drastically changing the initial price from \$74.27 to \$18.47 has very little effect on the kPA over all portfolio sizes. Changing the price to \$221.16 also has little effect on the results, as seen in Figure 29. Note that the price change is achieved by changing the initial short term deviation factor for TF, MR, and the fitted FS model. GBM and F are unchanged. In particular, as we use GBM to describe the long term behavior of oil price, it is unchanged since we do not alter the long term factor. This insensitivity to initial price gives us assurance that any model we use does not depend on one particular day's spot price.

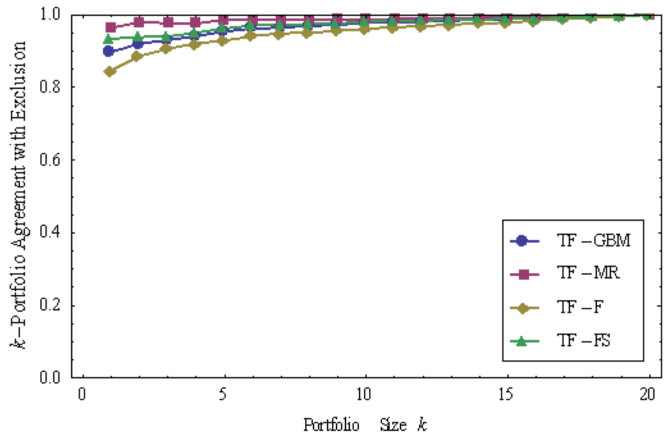

Figure 28: kPA using an initial price of $18.47.

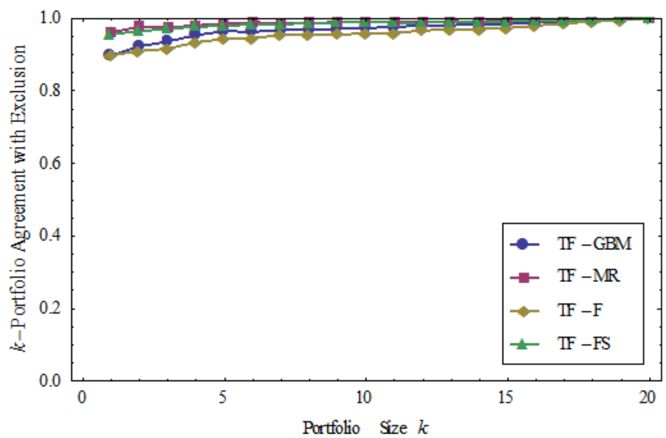

Figure 29: kPA using an initial price of $221.16.

5.2. Option

We now look at various situations in which the candidate projects have options. There is a wide variety of options we could consider: options to expand or contract production, increase or decrease our share of the project, delay production, prematurely end production, and temporarily increase or decrease production, to name a few. Other, more complex, options may allow a decision to be made after some specific sequence of events has happened. These are called "path dependent" options, and cannot be modeled with a recombining lattice, but can be modeled with a tree such as the binomial tree described by Brandão, Dyer, and Hahn (2005a). For problems with many time periods or many uncertainties, these trees can quickly become computationally intractable, and so we do not consider path dependent options in this thesis.

The possibilities for creating options are limitless, but we will restrain our analysis to a few instances that are not uncommon in industry. We could easily create a path dependent option that only TF would value correctly and say that this stochastic model is needed, however this scenario may not be realistic.

The literature has already considered some of the options mentioned above. Smith and McCardle (1998) analyze the value of a project with multiple options using both a GBM and MR model. They emphasize that modeling and analysis of options should focus on options that allow the decision maker to use knowledge gained over time. They model a number of options including delaying production, developing nearby fields, and cutting off production early. They found that under a mean reverting price model, the option to delay production had significantly lower value than under the random walk model. The time from the beginning of development to the start of production was long enough that short-term deviations in price had little effect on value. As we saw in the no-option analysis above, "the mean-reverting model suggests that the critical question is whether the project is profitable at long-run average prices; current prices are not particularly relevant" (Smith and McCardle 1998) . This agrees with our conclusions from the analysis of the sensitivity to initial price in the previous section.

Schwartz and Smith (2000) come to a similar conclusion using the TF model. They consider a long-term investment with significant setup time and a short-term option with no setup time. The short-term option value is sensitive to both long-term and short-term price factors, while the long-term investment is sensitive to only the long-term factor. They note that in analyzing long-term investments a one-factor model that incorporates only uncertainty in equilibrium prices (a GBM model) can substitute for the two-factor TF model. This agrees with our results that show GBM and TF as ranking projects almost identically.

This literature indicates that we should only model the price dynamics relevant to the decision, and that project and option value may be closely linked to our beliefs about price dynamics. Thus it is necessary and sufficient to model only the parameters relevant to valuation. We consider the question of whether the ability to make a future decision changes the impact of stochastic price models for ranking decisions.

5.2.1. A Simple Option

First, we look at an option from Brandão, Dyer, and Hahn (2005a) to buy out a partner's interest in a project or completely divest the company's interest, which we call the "buyout/divest" option. Suppose the set of projects we are choosing from for our portfolio are to be developed jointly with a partner firm, which owns a 25% interest, and we own the remainder. For each project individually, we have the option to buy out their 25% interest for \$40MM or sell our share for \$100MM at a single exercise date. This option has three alternatives: (1) buyout the partner, increasing our share by 1/3 for \$40MM, (2) divest for a lump sum of \$100MM, or (3) do nothing and let the option expire. The roll-back procedure described in §4.2 automatically calculates the value of (1), and the values of (2) and (3) are given.

Figure 30 shows the kPA for the same projects used in the previous sections, but with the addition of the buyout/divest option. The kPA with the addition of the option is still high; each model agrees with TF on over 90% of the portfolio composition on average for each *k*. The GBM kPA is over 99% for each *k* and with a kPA of 99.8% for *k*

= 1, only disagreed with TF on the best project in 1 out of 500 trials. The kPA for MR, F, and FS each slightly decreased after adding the option. Without the option, the three models each agreed with TF on the best project about 94% of the time, but with the option, F decreased to 93.4%, and MR and FS decreased further to 92.6% and 92.8%, respectively. Although slight, the option has some effect on project rankings. In a small number of cases, the differences in option valuations by the different models changed one models' project rankings, but did not change another's the same way.

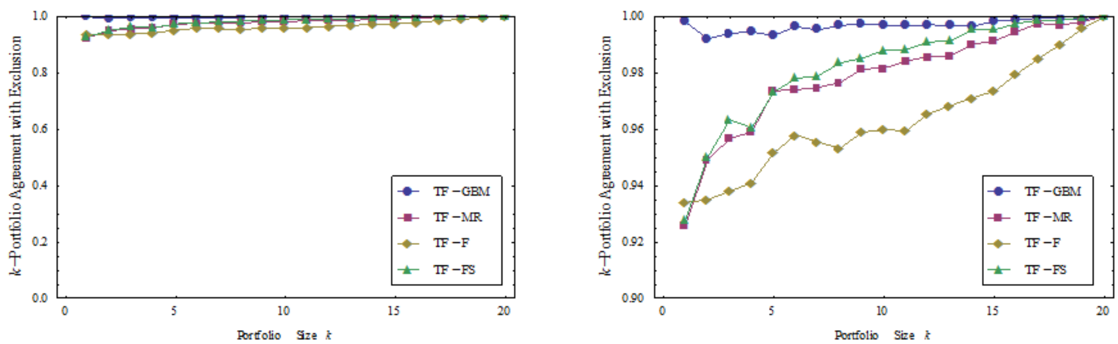

Figure 30: kPA for projects with buyout/divest option.

Figure 31 and Figure 32 show the kPAE and value lost for the projects with the buyout/divest option. As with the kPA, we see there is very little difference from the results for the projects without options.

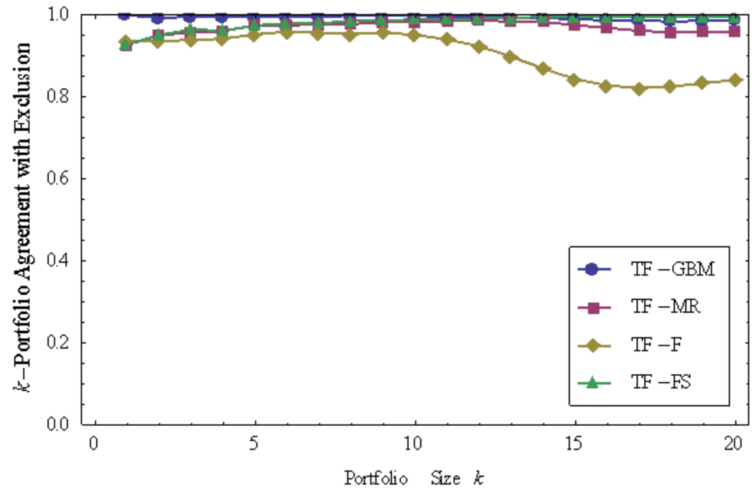

Figure 31: kPAE for projects with buyout/divest option.

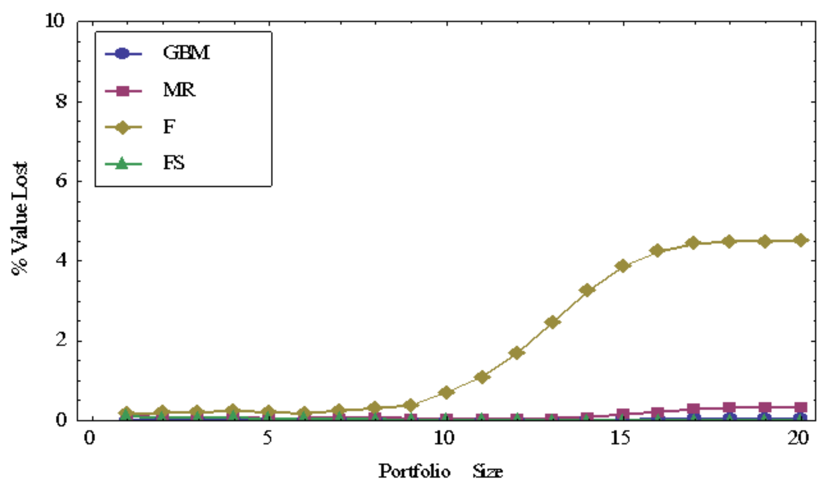

Figure 32: Value lost for projects with buyout/divest option.

The kPA and kPAE for projects with options is very similar to that for projects without the option. At first glance this may seem counterintuitive, because the projects now have options, whose values and optimal exercise are largely dependent on which price model is used. It would also seem that with the ability to make a future decision, a more accurate description of the uncertainty to be revealed should be relevant to portfolio composition. These issues drive the importance of stochastic price models for valuation-decisions, but have not significantly affected the ranking-decisions we consider here.

The option values are significant, with an average value of $181MM under TF, but the standard deviation of project values is about $665MM. Additionally, each project has the same option. All the projects are potentially increasing in value due to the option, but their values increase similarly. The options themselves simply do not have enough impact on project value, and do not differentiate between the projects enough (because they are all the same and the projects have similar production curve shapes) to affect decision making.

	Value Without Option ($MM)	Value of Option ($MM)	Value With Option ($MM)
TF	363.48	181.24	544.72
GBM	323.43	180.68	504.10
MR	541.44	145.48	686.29
F	100.87	90.93	191.80
FS	425.29	136.59	561.89

Table 10: Average project value with and without options, and average option value for each model.

Price models do not have significant relevance to ranking decisions without options; therefore in order for the models to have significant impact when the projects do have options, it must be the options that introduce some new discrimination between the projects. We now look at modifications to the scenario analyzed in this section to explore this idea.

5.2.2. Increasing the Buyout Value

The previous section demonstrated that the option values were not large enough to affect most project rankings. If we make the option more significant by doubling our stake in the project rather than increasing it by a third, will it make the options more important to the rankings? A realistic short-term option comparable to the one used by Schwartz and Smith (2000) would likely have a value very small relative to the size of the project, holding even less ability to affect rankings than the option considered above.

Figure 33 shows the kPA for each model with this more valuable option to double production. Table 11 shows the average value of this option for each model. The results still shows no significant difference, even though the option is on average about two to three times more valuable than before.

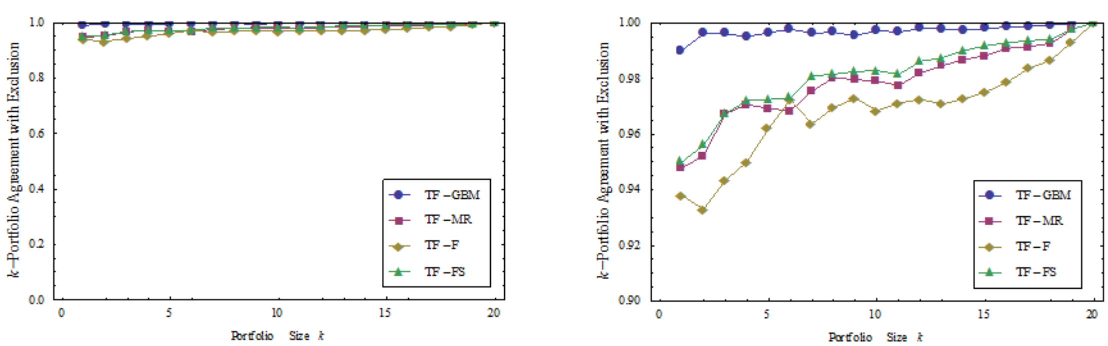

Figure 33: kPA for projects with options to double production.

Model	Original Option Value	Double-Production Option Value
TF	181.24	494.82
GBM	180.68	488.05
MR	145.48	461.05
F	90.93	237.94
FS	136.59	434.28

Table 11: Average option values for original buyout/divest option and the buyout/divest option to double production.

Increasing the option value by increasing the amount of project value it is worth does not discriminate the projects significantly more than the smaller option discussed earlier. Figure 34 shows a typical project production profile with the amount of production increase shown if we exercise the option. This applies to all projects, since they each have the same option. In valuing the option to expand separately from valuing the project itself, it is as if we are valuing an additional project with similar characteristics. When there was no option, the rankings by each of the models for the projects were very similar. If the options have similar profiles to projects, then we would expect to get the same result.

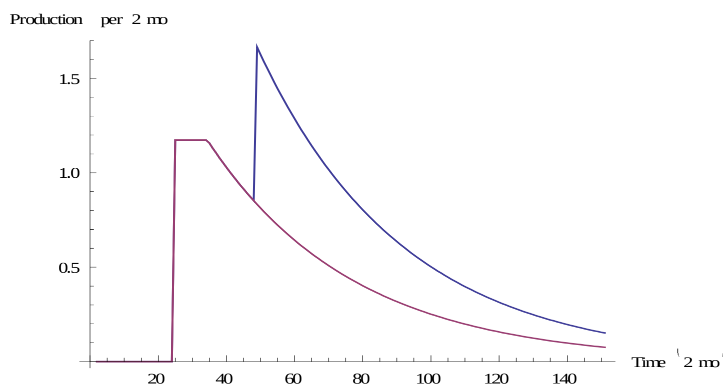

Figure 34: A typical project production profile showing the result of doubling our stake.

5.2.3. Similar Projects

Our candidate project sets are comprised of projects with widely varying values, such that the addition of options has little effect on their rankings. Simply increasing the "payoff" of the option does not change the results since the projects and options have similar characteristics. Here we consider a decision involving two very similar projects with the same option. However, due to the similarity of the projects, differing valuations of the option by different models discriminate the projects enough to change the decision.

Figure 35 shows the production profiles for two similar projects. F values them both at \$266.56MM even though the production profiles are not quite the same. Now suppose an option exists to buyout the partner's 1/3 stake for \$40MM at the beginning of year 10 for each project. This increases project A's value by \$58.7MM, but increases project B's by only \$40.56MM, making project A distinctly more attractive. The NPV of each project alone did not distinguish the projects, but the addition of identical options caused the total valuations to change, favoring one project over the other.

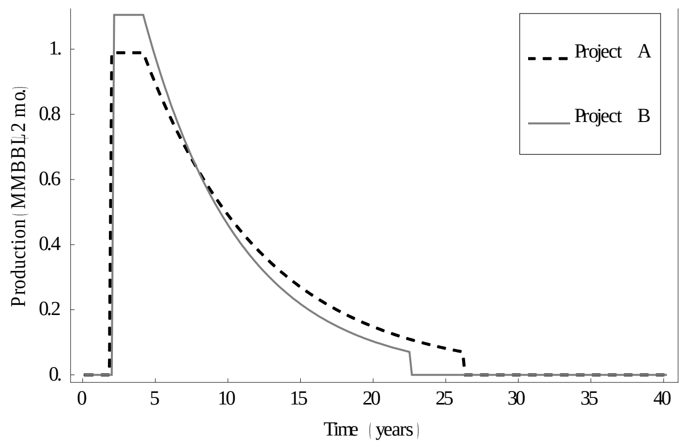

Figure 35: Production profiles of two similar projects.

Now suppose project B's peak production is increased by 5,000 bbl per two month period, which increases its project value by about $10MM, bringing it to $276.89MM. The value of the option on project B only increases by about 0.5MM. When using F, without the option, project B ($276.89MM) is preferred to project A ($266.56MM). With the option, project A ($325.26MM) is now preferred to project B ($317.93MM).

Figure 36 shows the values of A and B using a fixed price versus the cost to exercise the option. The rankings switch at an exercise cost of about $170MM. Above this value, using a fixed price will lead to incorrect rankings. Figure 37 shows the values of projects A and B using TF over a range of exercise costs, and A is always preferred to B. This result is highly dependent on correct valuation of the options. Since the projects have very similar values to begin with, any differences are very "fine-grained" and distinguishing the better project requires a fine-grained analysis using a stochastic model. However, since these projects have similar values and the option values are small, the impact of choosing the less valuable project is minor.

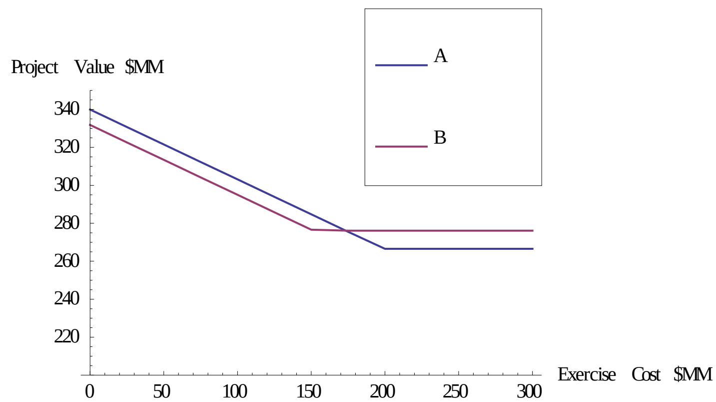

Figure 36: Total project value of the two projects using F as the exercise cost changes.

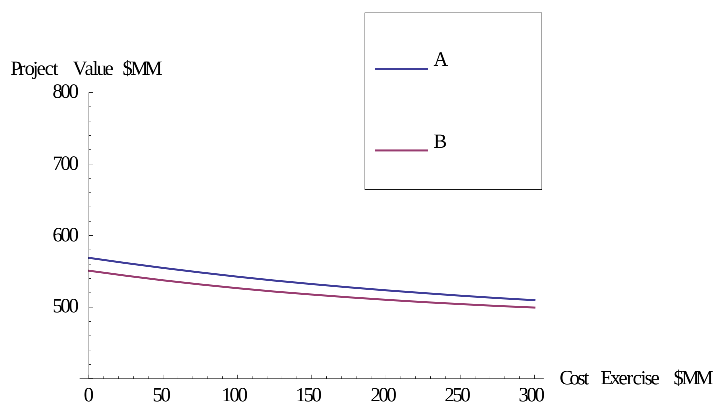

Figure 37: Total project value of the two projects using TF as the exercise cost changes.

5.2.4. Continuous-Exercise (American-Style) Option

The options we have considered up to now have all had a single exercise date. These options introduced only one future decision, which does not match the fidelity of a price model that models stochastic price changes in each time period. Increasing the number of decisions (to one per period) to match the fidelity of the price model (which can change each period) leads to an American-style option. In each time period, there is a decision to exercise the option or wait, up until either the option expires or is exercised. This new option can be exercised throughout the life of the project.

Figure 38 shows the kPA for projects with the American option. The kPA for each model and each portfolio size *k* are above 90%, although differences between the models are more apparent for small portfolios than with the single-exercise option. F agrees with TF on the top project 91.6% of the time, compared to 95.8% and 96.2% for MR and FS, respectively. GBM agrees with TF on the top project 99.8% of the time, and agrees with TF on over 99% on average for each portfolio size.

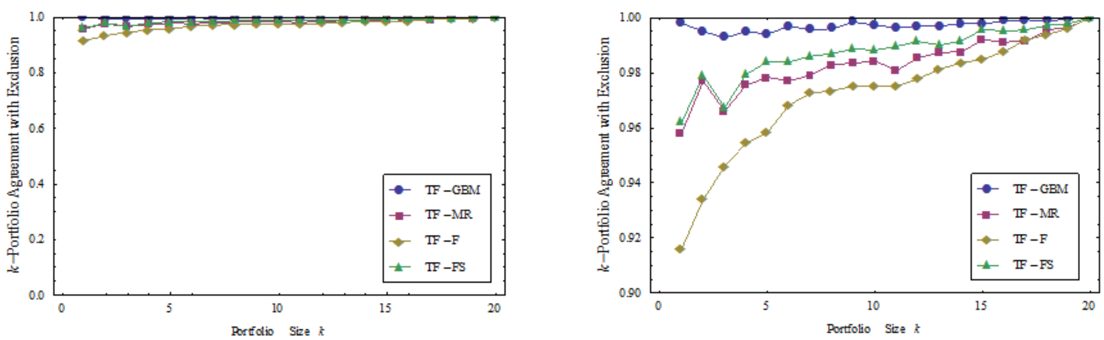

Figure 38: kPA for projects with American-style options.

Comparing these results to those from the original single-exercise buyout/divest option considered in §5.2.1, increasing the flexibility of the option, by increasing the number of exercise dates, has increased the kPA of MR and FS, while decreasing the kPA of F. Both MR and FS capture the long-term average behavior of oil price, but not the uncertainty. GBM models long-term behavior as well as uncertainty in the long-term price and has the highest kPA for all portfolio sizes. MR and FS have very similar kPAs and only model average long-term behavior without uncertainty. F models neither long-term behavior, nor uncertainty, and has the lowest kPA for each k less than 17. Above 17 it has approximately the same kPA as MR and FS.

These results indicate that the long-term behavior of oil price is the most important factor for a long-term project even when the option in a project is continuously exercisable. However, despite the decrease in the kPA of F, the impact on portfolio value is small. For $k < 10$, F leads to an average value lost of less than 0.1%. At $k = 15$, the average value lost for F reaches 1%, and achieves a maximum of only 2.2% when all 20 projects could potentially be chosen. Figure 39 shows the average value lost for each portfolio size k for each model.

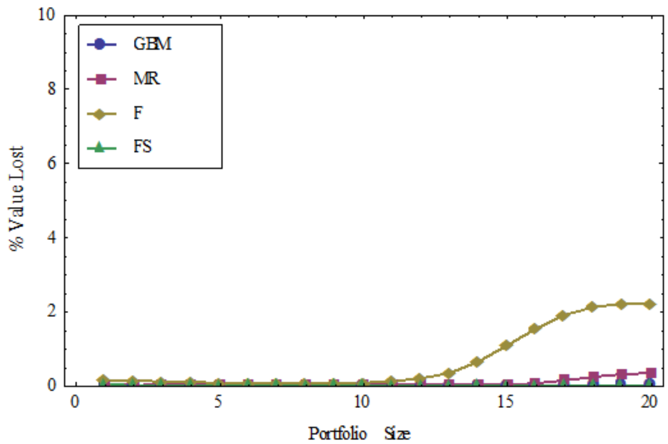

Figure 39: Value lost for each model with the American-style option.

6. Conclusions

Stochastic price models have proven material to decision making in the petroleum industry when accurate valuations are important, but as this thesis shows, are of little value when decisions depend on project rankings. It is well known, particularly in the real options literature, that stochastic price models are material to valuation-decisions. Ranking decisions have not been thoroughly examined, and the literature that discusses them does not consider significant project flexibility, or of impacts of uncertainty modeling on the values of portfolios constructed from rankings. We show that for ranking decisions in the form of oil project portfolio selection that stochastic price models have little impact, even when options are considered.

The relevance of stochastic price models is in a way proportional to the importance of accurate project valuations, or proportional to the accuracy required. We can speak in terms of the fidelity of the models and decisions. A high fidelity, or fine-grained, model can make more distinctions than a low fidelity model. A high fidelity decision exhibits many varying degrees of control in a system; it can be said to have a high degree of flexibility.

Modeling oil price as fixed removes a great deal of fidelity. In the example of the two similar projects in §5.2.3, F could not distinguish between the two projects without options. Reducing the model fidelity resulted in loss of distinction. However, in the portfolio composition simulations throughout Chapter 5, this reduction in distinctive power had very little effect on the ranking-decisions.

The question of the simplest model we can use is really a question of how much accuracy can we forfeit for ease of use without significantly impacting decision quality. The answer depends greatly on whether the decision is a valuation or ranking decision.

6.1. Valuation Decisions

If the decision requires accurate project valuation for good decision making, i.e. setting a selling price or determining how much to bid, then stochastic price models are material. The price models we consider yield not only differing valuations from a fixed price, but also from each other. Since accurate valuation is the focus of much of the real options literature, we do not give it much consideration.

Valuation can have an impact on ranking-decisions when projects are excluded on the basis of value. The magnitude of this impact depends on the size of the portfolio considered and the number of candidate projects that have positive value. If the portfolio size is small compared to the number of attractive projects, any lost value will be a result of incorrect rankings. When the portfolio size is about the same as the number of attractive projects the fact that certain models may systematically over- or under-estimate project values comes into play. A model that undervalues projects leads to investing fewer projects and passing up value, while a model that overestimates leads to choosing some projects that are expected to have negative value under TF. When the portfolio size is much larger than the number of attractive projects, the relative differences in the actual portfolios created with different models will be smaller, but value is still lost due to incorrect project valuations.

If an option is the deciding factor or tipping point in selecting a project or not, a stochastic model may be needed to correctly determine project rank. We look at a simple case where selecting the better of two possible projects can change depending on whether a stochastic model or a fixed price is used. In that case, addition of the option changed the projects' relative ranks and the incorrect valuation by F lead to an inferior decision of which to choose.

6.2. RANKING DECISIONS

Ranking-decisions are not commonly discussed in the literature, but appear in real-world situations. Budgetary limits are important, but are not the only constraints in choosing projects. Limits on available equipment or personnel, technological constraints, or contractual agreements for example, could place more stringent restrictions on how many projects to undertake than the available budget. In this case, we choose as many of the most valuable projects as the constraints allow.

Stochastic price models have little impact on the composition of portfolios of oil projects when only ranks are considered. If there are no negative-valued projects, or those projects have already been culled from the candidate set, only a ranking of the candidate projects is required to make a decision. Almost identical decisions are made when a well-estimated fixed price is used instead of the complex TF stochastic model. Adding options to the projects also has little effect. Although the value of each project changes, the rankings are largely unaffected. This is significant, as we have identified situations with significant options where stochastic models are not important. The presence of options does not necessitate stochastic modeling; the characteristics of the decision itself should also be considered.

When there are projects that may have negative values, the decision of portfolio composition is not purely ranking or valuation, but instead incorporates some degree of both. Projects are initially selected based on their ranks, and then possibly excluded from the portfolio based on their values. The exact value of a project is important only as far as determining whether that project has positive or negative value. In this situation, the choice of model for oil price has some small but noticeable impact on portfolio value depending on the portfolio's size. Rankings are model-independent, but incorporating exclusion leads to incorporating project valuations, which are highly model-dependent.

6.3. Questions to Ask

The impact of modeling choice depends on the decision itself and characteristics of the projects. As such, we cannot make a definitive statement of when to model price stochastically. However, there are some guiding questions to ask that can illuminate the particular situation decision-makers face.

Are rankings or valuations more important to the decision?

In other words, do we face a ranking-decision or a valuation-decision? If it is a valuation decision, we immediately know to use stochastic models. If it is a ranking-decision, stochastic models cannot be immediately discounted, and further investigation is necessary.

How similar are the projects?

If the projects are very similar, a high fidelity model may be required to achieve an adequate level of distinction. However, if the projects are similar in many characteristics so that the production profiles are almost identical, the value lost by choosing inferior projects will be inherently small. It is unrealistic to assume two separate projects are identical, but if production is modeled deterministically, and there is no specific reason to believe the projects are different, we face this exact situation. If after a simple preliminary valuation using a fixed price, some projects have nearly identical values but their production profiles are known to be different, use of a stochastic model may be warranted. Portfolios of projects that are clearly very different are less dependent on stochastic modeling of oil price if there are more positive-valued projects than can be chosen for the portfolio.

What is the relationship between the options and the projects?

Options with more flexibility require greater model fidelity in the factors important to the option. Schwartz and Smith (2000) displayed this with their example of a short-term option and a long-term option, each of which depended on different factors in the TF model. Options with values small relative to the project values are unlikely to change project rankings if accurately valued, indicating that stochastic modeling is not necessary. The question essentially seeks to determine whether options represent significant portions of project value.

6.4. Application

How can these results be useful in practice? When might further analysis of price uncertainty be warranted and how will we know? For the case of ranking-decisions, the results suggest a two-step analysis. The first step is to value the projects using a well-estimated fixed price. If there are more positive-valued projects than can be included in the portfolio, then the fixed price is sufficient to build a portfolio with value likely to be very close to the optimal portfolio value chosen using a stochastic model. If there are fewer positive value project than can be included, then proceed to the second step of using a stochastic price model, mindful of what price process characteristics are relevant to the decision.

This two-step procedure is related to the idea of sensitivity analysis. In determining how much fidelity to build into the model of uncertainty, it is important to consider how sensitive the decision is to that uncertainty. In our portfolio selection ranking decision, when the values of some projects are near zero, the decision of portfolio composition may be very sensitive to how uncertain we are of oil prices. When the values are very large, high uncertainty in price may have little or no effect on the decision.

6.5. Extensions and Future Work

There are several possible avenues to explore further. We made several assumptions to arrive at our conclusions. A more complete project model would include production and cost uncertainties, although real options theory says we can separately handle diversifiable (oil price) and non-diversifiable (production) uncertainties. Risk aversion would factor into the problem for the non-diversifiable uncertainties, but analysis of oil price uncertainty would remain the same.

Another avenue is to consider nonlinear profit functions caused by tax or royalty structures, for example. There are many possible scenarios to consider, and the particular structure of the profit function would be relevant to the choice of price model as it may amplify or attenuate prices in certain ranges.

Here we only consider scale options, and identical options for all projects. Portfolios of differing projects with significant and differing levels of flexibility may be more dependent on stochastic modeling than the situations we consider. We also consider one kind of cash-flow (production) profile, which is common in the petroleum industry, but cash-flow characteristics may differ greatly from project to project, or in projects in other industries.

We also touch on the use of a three-point discretization of price, but do not examine its use thoroughly. The discretization improved slightly over a simple fixed price, but an interesting question is how much improvement would be gained if the "correct" discretization were used, and what that dsicretization would be. It is not clear how to handle uncertain quantities that change with time in a single discretization.

These points for further exploration represent many issues encountered in carrying out this work. Other interesting questions certainly exist that further detail, as well as broaden, the conclusions presented here.

Appendix: Grid Probability Calculation

Mathematica 8.0 code for calculating transition probabilities for the two-factor model grid approximation:

Inputs

- deltat - size of time step in years
- devsteps - number of short-term factor states
- eqsteps - number of long-term factor states
- kappa - mean reversion factor
- sigma1 - short-term factor standard deviation
- xbar - short-term factor mean
- mu - long-term factor drift
- sigma2 - long-term factor standard deviation
- rho - correlation factor

Outputs

- tmatrix - transition probability matrix
- states - 2-row matrix containing the long-term and short-term state values

```
GenerateTF[deltat_,devSteps_,eqSteps_,kappa_,sigma1_,xbar_,mu_,si
gma2_,rho_]:=
 (

  (*factors calculated from parameters*)
  stepsigma1=sigma1*Sqrt[(1-Exp[-2*kappa*deltat])/(2*kappa)];
  stepsigma2=sigma2*Sqrt[deltat];
  regcoeff=rho*stepsigma1/stepsigma2;
  conditionalsd=Sqrt[stepsigma1^2-(stepsigma2*regcoeff)^2];
  (*parameters of price model factor states*)
  devSmallState=-1.3;
  devBigState=1.3;
  eqSmallState=2;
  eqBigState=6.5;
```

```
  devStepSize=(devBigState-devSmallState)/(devSteps-1);
  eqStepSize=(eqBigState-eqSmallState)/(eqSteps-1);

  devStateValues=ConstantArray[0,devSteps];
  devLowerCutoff=ConstantArray[0,devSteps];
  devUpperCutoff=ConstantArray[0,devSteps];
  devStateValues[[1]]=devSmallState;
  devLowerCutoff[[1]]=-500;
  devUpperCutoff[[devSteps]]=500;

  For[i=2,i□devSteps,i++,
   (
    devStateValues[[i]]=devStateValues[[i-1]]+devStepSize;
    devUpperCutoff[[i-1]]=(devStateValues[[i-
1]]+devStateValues[[i]])/2;
    devLowerCutoff[[i]]=devUpperCutoff[[i-1]];
    )];

  eqStateValues=ConstantArray[0,eqSteps];
  eqLowerCutoff=ConstantArray[0,eqSteps];
  eqUpperCutoff=ConstantArray[0,eqSteps];
  eqStateValues[[1]]=eqSmallState;
  eqLowerCutoff[[1]]=-500;
  eqUpperCutoff[[eqSteps]]=500;

  For[i=2,i□eqSteps,i++,
   (
    eqStateValues[[i]]=eqStateValues[[i-1]]+eqStepSize;
    eqUpperCutoff[[i-1]]=(eqStateValues[[i-
1]]+eqStateValues[[i]])/2;
    eqLowerCutoff[[i]]=eqUpperCutoff[[i-1]];
    )];

  (*store all eq and dev state combinations in a 2D array*)
  states=ConstantArray[0,{2,devSteps*eqSteps}];
  For[i=0,i<devSteps*eqSteps,i++,
   (
    (*first row is eq states*)
    states[[1,i+1]]=eqStateValues[[Floor[i/devSteps]+1]];
    (*second row is dev states*)
    states[[2,i+1]]=devStateValues[[Mod[i,devSteps]+1]];
    )];

  (*matrix of conditional means*)
  devCMeans=ConstantArray[0,{eqSteps*devSteps,eqSteps}];
  For[i=0,i<eqSteps*devSteps,i++,
   (
```

```
    For[j=1,j□eqSteps,j++,
      (

devCMeans[[i+1,j]]=xbar+(devStateValues[[Mod[i,devSteps]+1]]-
xbar)*Exp[-kappa*deltat]+regcoeff*(eqStateValues[[j]]-
eqStateValues[[Floor[i/devSteps]+1]])
       )];
    )];
  eqMeans=states[[1,All]]+mu*deltat;
  tmatrix=ConstantArray[0,{devSteps*eqSteps,devSteps*eqSteps}];

  For[i=1,i□devSteps*eqSteps,i++,
   (
    For[j=0,j<devSteps*eqSteps,j++,
      (
       tmatrix[[i,j+1]]=(CDF[NormalDistribution[

devCMeans[[i,Floor[j/devSteps]+1]],conditionalsd],devUpperCutoff[
[Mod[j,devSteps]+1]]]-
CDF[NormalDistribution[devCMeans[[i,Floor[j/devSteps]+1]],conditi
onalsd],devLowerCutoff[[Mod[j,devSteps]+1]]])*(CDF[NormalDistribu
tion[eqMeans[[i]],stepsigma2],eqUpperCutoff[[Floor[j/devSteps]+1]
]]-
CDF[NormalDistribution[eqMeans[[i]],stepsigma2],eqLowerCutoff[[Fl
oor[j/devSteps]+1]]]);
       )];
    )];

  matdim=devSteps*eqSteps;
  mat=ConstantArray[0,{matdim,matdim}];
  For[i=1,i□matdim,i++,
   (
    For[j=1,j□matdim,j++,
      (

If[tmatrix[[i,j]]<0.00001,mat[[i,j]]=0,mat[[i,j]]=tmatrix[[i,j]]]
;
       )];
    )];
  tmatrix=SparseArray[mat];

        {tmatrix,states}
```

Code for calculating project values and option values:

Inputs

- prodrate - array of production in each period in millions of barrels
- prodcost - array of costs incurred in each period
- tmatrix and states calculated with GenerateTF code above
- exercisecost - cost to exercise opton
- index - column index of states matrix indicating the initial state
- deltat - length of time period
- years - the number of years to consider
- optiontimes - set of times (in number of periods) when the option can be exercised
- annualdiscountrate - the annual discount factor

Outputs

- three element array containing the total value, the project value, and the option value

```
      solvegrid[prodrate_, prodcost_, tmatrix_, states_,
exercisecost_, index_, deltat_, years_, optiontimes_,
annualdiscountrate_] :=
      (
      timesteps = years/deltat;
      termtime = timesteps;
      For[t = timesteps, t > 1, t--,
      If[prodrate[[t]] == 0 && prodrate[[t - 1]] > 0, termtime =
t, {}]];
      timesteps = termtime;
      numstates = Length[states[[1, All]]];
      dfactor = Exp[-annualdiscountrate*deltat];
      termarray = ConstantArray[0, {Length[states[[1, All]]]}];
      vRecurse[x_,
      t_] := (prodrate[[t]]*Exp[states[[1, All]] + states[[2,
All]]] -
```

```
      ConstantArray[prodcost[[t]], numstates] +
dfactor*tmatrix.x);
      valuematrix =
      MapThread[
      Reverse, {Transpose[
      FoldList[vRecurse, termarray, Reverse[Range[timesteps +
1]]]]]}];
      optionmatrix =
      MapThread[
      Append, {ConstantArray[0, {numstates, timesteps + 1}],
      ConstantArray[0, numstates]}];
      For[t = timesteps, t >= 0, t--,
      (
      If[MemberQ[optiontimes, t],
      optionmatrix[[All, t + 1]] = MapThread[Max, {
      (1/3)*valuematrix[[All, t + 1]] - ConstantArray[40,
numstates],
      ConstantArray[100, numstates],
      dfactor*tmatrix.optionmatrix[[All, t + 2]]
      }],
      optionmatrix[[All, t + 1]] =
      dfactor*tmatrix.optionmatrix[[All, t + 2]]];
      )];
      optionvalue = optionmatrix[[index, 1]];
      basevalue = valuematrix[[index, 1]];
      {optionvalue + basevalue, basevalue, optionvalue}
      )
```

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